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## CLOSURE OPERATORS ON COMPLETE ALMOST DISTRIBUTIVE LATTICES-III<sup>†</sup>

### Abstract

In this paper, we prove that the lattice of all closure operators of a complete Almost Distributive Lattice  $L$  with fixed maximal element  $m$  is dual atomistic. We define the concept of a completely meet-irreducible element in a complete ADL and derive a necessary and sufficient condition for a dual atom of  $\Phi(L)$  to be complemented.

*Keywords:* Complete Almost Distributive Lattice, Closure operator, Dual atom, Dual atomistic, Completely meet-irreducible element .

### 1. Introduction:

In [17] Swamy and Rao introduced the concept of an Almost Distributive Lattice (ADL) as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra like  $p$ -rings [12], regular rings [11], biregular rings [16], associate rings [10],  $p_1$ -rings [13], triple systems [15], baer rings [1],  $m$ -domain rings [14] and  $*$ -rings [2] on one hand and the class of distributive lattices on the other. Thus, a study of any concept in the class of ADLs will yield results in all the classes of algebras mentioned above. In [17], they also observed that the set  $PI(L)$  of all principal ideals of an ADL  $(L, \vee, \wedge, 0, m)$  with a maximal element  $m$ , forms a bounded

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distributive lattice. Through this distributive lattice  $PI(L)$ , many existing concepts of lattice theory were extended to the class of ADLs [3, 4, 5, 18].

In mathematics, closure operators play important role in topology, algebra and logic and in theoretical computer sciences, closure operators have been widely used in the semantic area, notably in domain theory, in program semantics and in the theory of semantics approximation by abstract interpretation. In view of the rich applications of complete lattices and the closure operators in different fields, we introduced the concept of a complete ADL  $L$  in [6] and the concept of a closure operator of a complete ADL in [7, 8] and derived some important properties on closure operators. In this paper, we define the concept of a completely meet irreducible element in a complete ADL  $(L, \vee, \wedge, 0, m)$  and establish a relation between completely meet irreducible elements in a complete ADL  $L$  and dual atoms of the lattice  $(\Phi(L), \leq)$  of all closure operators of  $L$ . We derive necessary and sufficient conditions for dual atoms in the lattice  $(\Phi(L), \leq)$  to have complements.

## 2. Preliminaries

DEFINITION 2.1. [17] An algebra  $(L, \vee, \wedge, 0)$  of type  $(2, 2, 0)$  is called an Almost distributive lattice (ADL) if, for any  $a, b, c \in L$ , the following conditions hold:

- |  |                                 |
|--|---------------------------------|
| (1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ . | (4) $(a \vee b) \wedge b = b$ . |
| (2) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ . | (5) $a \vee (a \wedge b) = a$ . |
| (3) $(a \vee b) \wedge a = a$ .                              | (6) $0 \wedge a = 0$ .          |

LEMMA 2.2. [17] If  $(L, \vee, \wedge, 0)$  is an ADL, for any  $a, b \in L$ , define  $a \leq b$  if and only if  $a = a \wedge b$  or equivalently  $a \vee b = b$ , then  $\leq$  is a partial ordering on  $L$  and for any  $a, b, c \in L$ , we have the following:

- (1)  $a \vee b = a \Leftrightarrow a \wedge b = b$
- (2)  $a \vee b = b \Leftrightarrow a \wedge b = a$
- (3)  $a \vee b = b \vee a$  whenever  $a \leq b$
- (4)  $\wedge$  is associative in  $L$
- (5)  $a \wedge b \wedge c = b \wedge a \wedge c$
- (6)  $(a \vee b) \wedge c = (b \vee a) \wedge c$
- (7)  $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
- (8)  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- (9)  $a \wedge (a \vee b) = a$ ,  $(a \wedge b) \vee b = b$  and  $a \vee (b \wedge a) = a$

- (10)  $a \leq a \vee b$  and  $a \wedge b \leq b$
- (11)  $a \wedge a = a$  and  $a \vee a = a$
- (12)  $0 \vee a = a$  and  $a \wedge 0 = 0$
- (13) If  $a \leq c, b \leq c$ , then  $a \wedge b = b \wedge a$  and  $a \vee b = b \vee a$
- (14)  $a \vee b = (a \vee b) \vee a$ .

In the above Lemma, properties (5) and (6) are restricted commutativity of  $\vee$  and  $\wedge$  respectively. From this we get that , for any  $a, b \in L$ , the interval  $[a, b] := \{x \in L \mid a \leq x \leq b\}$  is a bounded distributive lattice.

**THEOREM 2.3.** [17] *Let  $(L, \vee, \wedge, 0)$  be an ADL. Then, for any  $m \in L$ , the following are equivalent:*

- (1)  $m$  is maximal
- (2)  $m \vee x = m$  for all  $x \in L$
- (3)  $m \wedge x = x$  for all  $x \in L$ .

If  $(L, \vee, \wedge, 0)$  is an ADL and  $m$  is a fixed maximal element of  $L$ , then we say that  $(L, \vee, \wedge, 0, m)$  is an **ADL with a maximal element  $m$** .

**DEFINITION 2.4.** [17] A non empty subset  $I$  of an ADL  $L$  is said to be an ideal of  $L$ , if  $a \vee b, a \vee x \in I$  for all  $a, b \in I, x \in L$ .

For any non empty subset  $S$  of  $L$

$$(S) = \{(\bigvee_{i=1}^n s_i) \wedge x \mid s_i \in S, x \in L, n \text{ is a positive integer}\}$$

is the smallest ideal of  $L$  containing  $S$ . In particular, for any  $x \in L, (x) = (\{x\}) = \{x \wedge t \mid t \in L\}$  and  $(x)$  is called the principal ideal generated by  $x$ . The set  $I(L)$  of all ideals of  $L$  is closed under arbitrary intersection and contains  $L$ . Thus  $(I(L), \vee, \wedge)$  is a complete lattice where  $I \vee J = \{x \vee y \mid x \in I, y \in J\}$  and  $I \wedge J = I \cap J$  for any  $I, J \in I(L)$ . Since, for any  $x, y \in L, (x) \vee (y) = (x \vee y)$  and  $(x) \wedge (y) = (x \wedge y)$ , the set  $PI(L)$  of all principal ideals of  $L$  is a sublattice of  $I(L)$ . The lattice  $PI(L)$  plays a very important role in the development of the theory of ADLs. One can extended many existing concepts from the class of distributive lattices to the class of ADLs though this class of principal ideals.

DEFINITION 2.5. [6] An ADL  $L = (L, \vee, \wedge, 0, m)$  with a maximal element  $m$  is called a complete ADL if  $PI(L)$  is a complete sublattice of  $I(L)$ , or equivalently,  $[0, m]$  is a complete distributive lattice.

DEFINITION 2.6. [7] Let  $L$  be a complete ADL with a maximal element  $m$ . Then a mapping  $\phi : L \rightarrow L$  is said to be a closure operator of  $L$  if, for any  $x, y \in L$ , the following conditions hold:

- (1)  $\phi(x) \leq m$
- (2)  $\phi(x) \wedge x = x$
- (3) If  $x \leq y$ , then  $\phi(x) \leq \phi(y)$
- (4)  $\phi(x \wedge y) = \phi(y \wedge x)$
- (5)  $\phi(\phi(x)) = \phi(x)$ .

DEFINITION 2.7. [7] Let  $L$  be a complete ADL with a maximal element  $m$ , and  $\phi$  a closure operator of  $L$ . Then an element  $x \in L$  said to be closed under  $\phi$ , if  $\phi(x) = x$ . Clearly,  $m$  is closed under every closure operator of  $L$ .

LEMMA 2.8. [7] Let  $L$  be a complete ADL with a maximal element  $m$ , let  $\phi$  be a closure operator of  $L$  and  $\{x_\alpha \mid \alpha \in J\}$  a family of elements of  $L$  closed under  $\phi$  in  $L$ . Then  $\bigwedge_{\alpha \in J} (x_\alpha)$  is also an element of  $L$  closed under  $\phi$  in  $L$ .

If we define  $t$  and  $\omega : L \rightarrow L$  by  $t(x) = x \wedge m$  and  $\omega(x) = m$  for all  $x \in L$ , then  $t, \omega$  are closure operators of  $L$

THEOREM 2.9. [7] Let  $\Phi(L)$  be the set of all closure operators of  $L$  and for any  $\phi, \psi \in \Phi(L)$ , define  $\phi \leq \psi$  if and only if  $\phi(x) \leq \psi(x)$  for all  $x \in L$ . Then  $(\Phi(L), \leq)$  is a complete lattice in which the greatest element is  $\omega$  and least element is  $t$ .

LEMMA 2.10. [7] Let  $L$  be a complete ADL with a maximal element  $m$ ,  $a \in L$  such that  $a \wedge m \neq m$  and define  $\phi_a : L \rightarrow L$  by  $\phi_a(x) = a \wedge m$ , if  $a \wedge x = x$  and  $\phi_a(x) = m$ , if  $a \wedge x \neq x$  for all  $x \in L$ , then  $\phi_a$  is a closure operator of  $L$ .

**THEOREM 2.11.** [7] *Let  $L$  be a complete ADL with a maximal element  $m$  and  $\Phi(L)$  be the set of all closure operators of  $L$ . Then we have the following:*

- (1) *If  $\{\phi_\alpha \mid \alpha \in J\} \subseteq \Phi(L)$  and  $\psi = \bigvee_{\alpha \in J} \phi_\alpha$ , then, for any  $x \in L$ ,  $\psi(x) = x$  if and only if  $\phi_\alpha(x) = x$  for all  $\alpha \in J$ .*
- (2) *If  $a \in L$  such that  $a \wedge m \neq m$ , then  $\phi_a$  is a dual atom of  $\Phi(L)$ .*
- (3) *Every dual atom of  $\Phi(L)$  is of the form  $\phi_b$  for some  $b \in L$  such that  $b < m$ .*
- (4) *For  $\phi_1, \phi_2 \in \Phi(L)$ ,  $\phi_1 \leq \phi_2$  if and only if, for any  $x \in L$ ,  $\phi_2(x) = x$  implies  $\phi_1(x) = x$ .*

For all standard definitions and results in distributive lattices we refer to Gratzner, G. [9].

### 3. Complemented Closure Operators

In this section, we deal with the dual atoms of the lattice  $(\Phi(L), \leq)$ , where  $\Phi(L)$  is the set of all closure operators of a complete ADL  $L$ . We define the concept of a completely meet-irreducible element in  $L$  and we prove a necessary and sufficient conditions for a dual atom  $\phi_a$  (where  $a \in L$  such that  $a \wedge m \neq m$ ) to have a complement in the lattice  $(\Phi(L), \leq)$ .

We begin this section with the following Definition

**DEFINITION 3.1.** Let  $L$  be complete ADL with a maximal element  $m$  and  $\phi$  a closure operator of  $L$ . Define  $F_\phi = \{x \in L \mid \phi(x) = x\}$ . That is,  $F_\phi$  is the set of elements of  $L$  closed under  $\phi$ .

**LEMMA 3.2.** *Let  $L$  be a complete ADL with a maximal element  $m$ ,  $M(\neq \emptyset) \subseteq [0, m]$  such that  $\text{Inf } M' \in M$  for all  $M' \subseteq M$  and for each  $x \in L$ , define  $\phi : L \rightarrow L$  by  $\phi(x) = \text{Inf } A_x$ , where  $A_x = \{y \in M \mid y \wedge x = x\}$ . Then  $\phi$  is a closure operator of  $L$ .*

**PROOF:** Now, we prove that  $\phi$  is a closure operator of  $L$ .

- (1) Clearly,  $\phi(x) \leq m$  for all  $x \in L$ .
- (2) By our assumption, we get that  $\phi(x) \in M$  for all  $x \in L$  and hence  $\phi(x) \wedge x = x$ .

- (3) Let  $x_1, x_2 \in L$  such that  $x_1 \leq x_2$ . Let  $A_x = \{y \in M \mid y \wedge x = x\}$ . Suppose  $y \in A_{x_2}$ . Then  $y \wedge x_2 = x_2$  and hence  $y \wedge x_1 = y \wedge x_2 \wedge x_1 = x_2 \wedge x_1 = x_1$ . Hence  $y \in A_{x_1}$ . Therefore  $A_{x_2} \subseteq A_{x_1}$ . Thus  $\inf A_{x_1} \leq \inf A_{x_2}$ . Hence  $\phi(x_1) \leq \phi(x_2)$ .
- (4) Let  $z \in A_{x \wedge y}$ . Then  $z \wedge x \wedge y = x \wedge y$  and hence  $z \wedge x \wedge y \wedge x = x \wedge y \wedge x$ . Therefore  $z \wedge y \wedge x = y \wedge x$ . Hence  $z \in A_{y \wedge x}$ . Thus  $A_{x \wedge y} \subseteq A_{y \wedge x}$ . By symmetry, we get that  $A_{y \wedge x} \subseteq A_{x \wedge y}$ . Therefore  $A_{x \wedge y} = A_{y \wedge x}$ . Hence  $\phi(x \wedge y) = \phi(y \wedge x)$ .
- (5) Since  $\phi(x) \wedge x = x$ , we get that  $x \wedge m \leq \phi(x)$  and hence  $\phi(x) \leq \phi(\phi(x))$ . We have,  $\phi(x) \wedge \phi(x) = \phi(x)$ , we get that  $\phi(x) \in A_{\phi(x)}$ . Hence  $\inf A_{\phi(x)} \leq \phi(x)$ . Therefore  $\phi(\phi(x)) \leq \phi(x)$ . Thus  $\phi(\phi(x)) = \phi(x)$ . Therefore  $\phi$  is a closure operator of  $L$ .  $\square$

Now, we prove the following Theorem

**THEOREM 3.3.** *Let  $L$  be a complete ADL with a maximal element  $m$  and  $M (\neq \emptyset) \subseteq [0, m]$ . Then there is a closure operator  $\phi$  of  $L$  such that  $M = F_\phi$  if and only if  $\text{Inf } M' \in M$  for all  $M' \subseteq M$ .*

**PROOF:** Suppose  $\phi$  is a closure operator of  $L$  and  $M = F_\phi$ . Let  $M' \subseteq M$ . Suppose  $x = \text{Inf } M'$ . Since every element of  $M'$  is closed under  $\phi$  and by Lemma 2.8, the infimum of closed elements under  $\phi$  is again closed under  $\phi$ , we get that  $x \in M$ . Conversely, suppose that  $\text{Inf } M' \in M$  for all  $M' \subseteq M$ . Now, we prove that there exists a closure operator  $\phi$  of  $L$  such that  $M = F_\phi$ . For each  $x \in L$ , define  $\phi : L \rightarrow L$  by  $\phi(x) = \text{Inf } \{y \in M \mid y \wedge x = x\}$ . Then by Lemma 3.2, we get that  $\phi$  is a closure operator of  $L$ . Let  $x \in F_\phi$ . Then  $\phi(x) = x$ . Now,  $\{y \in M \mid y \wedge x = x\} \subseteq M$  implies that  $\phi(x) = \text{Inf } \{y \in M \mid y \wedge x = x\} \in M$ , by our assumption. Thus  $\phi(x) \in M$ . That is,  $x \in M$ . Hence  $F_\phi \subseteq M$ . Now, suppose  $x \in M$ . So that  $\phi(x) \leq x$ . Thus  $\phi(x) = \phi(x) \wedge x = x$ . Therefore  $x \in F_\phi$ . Hence  $M \subseteq F_\phi$ . Thus  $M = F_\phi$ .  $\square$

**LEMMA 3.4.** *Let  $L$  be a complete ADL with a maximal element  $m$ . Let  $\phi, \psi \in \Phi(L)$ . Then  $\phi \leq \psi$  if and only if  $F_\psi \subseteq F_\phi$ .*

**PROOF:** Let  $\phi, \psi \in \Phi(L)$ . Suppose  $\phi \leq \psi$ . Then  $\phi(x) \leq \psi(x)$  for each  $x \in L$ . Let  $x \in F_\psi$ . Then  $\psi(x) = x$  and hence  $\phi(x) \leq x$ . So that  $\phi(x) = \phi(x) \wedge x = x$ . Therefore  $x \in F_\phi$ . Thus  $F_\psi \subseteq F_\phi$ . Conversely,

suppose that  $F_\psi \subseteq F_\phi$ . Now, we prove that  $\phi \leq \psi$ . Let  $x \in L$  such that  $\psi(x) = x$ . Then  $x \in F_\psi$ . Hence  $x \in F_\phi$ . Therefore  $\phi(x) = x$ . Thus  $\phi \leq \psi$ , by Theorem 2.11(4).  $\square$

LEMMA 3.5. *Let  $L$  be a complete ADL with a maximal element  $m$ . Let  $\{\phi_\alpha \mid \alpha \in J\} \subseteq \Phi(L)$ . Then  $F \bigvee_{\alpha \in J} \phi_\alpha = \bigcap_{\alpha \in J} F_{\phi_\alpha}$*

PROOF: Let  $x \in L$ . Then  $x \in F \bigvee_{\alpha \in J} \phi_\alpha \Leftrightarrow (\bigvee_{\alpha \in J} \phi_\alpha)(x) = x \Leftrightarrow \phi_\alpha(x) = x$  for all  $\alpha \in J$  (by Theorem 2.11(1))  $\Leftrightarrow x \in F_{\phi_\alpha}$  for all  $\alpha \in J \Leftrightarrow x \in \bigcap_{\alpha \in J} F_{\phi_\alpha}$ .

Thus  $F \bigvee_{\alpha \in J} \phi_\alpha = \bigcap_{\alpha \in J} F_{\phi_\alpha}$ .  $\square$

DEFINITION 3.6. Let  $X$  be a complete lattice. An element  $a \in X$  is said to be dual atomistic, if it is the infimum of set of all dual atoms above it.

DEFINITION 3.7. A closure operator  $\phi$  of a complete ADL  $L$  is called dual atom if  $\phi \leq \psi \leq \omega$  for any closure operator  $\psi$  of  $L$ , then either  $\psi = \phi$  or  $\psi = \omega$ .

Now, we prove the following Theorem.

THEOREM 3.8. *Let  $L$  be a complete ADL with a maximal element  $m$ . Then the lattice  $(\Phi(L), \vee, \wedge)$  is dual atomistic.*

PROOF: Let  $\psi \in \Phi(L)$  and  $\psi \neq \omega$ . Write  $A_\psi = \{x \in L \mid \psi(x) = x \text{ and } x \neq m\}$ . Choose  $y \in L$  such that  $\psi(y) \neq m$ . Write  $x = \psi(y)$ . Then  $\psi(x) = x$  and  $x \neq m$ . Therefore  $x \in A_\psi$ . Hence  $A_\psi \neq \emptyset$ . Also, by Lemma 2.11(2),  $\phi_x$  is a dual atom of  $\Phi(L)$  for all  $x \in A_\psi$ . Now, we prove that  $\psi = \bigwedge_{x \in A_\psi} \phi_x$ .

Let  $x \in A_\psi$  and  $y \in L$  such that  $\phi_x(y) = y$  and hence  $y = x(x \wedge m = x)$  or  $y = m$ . If  $y = x$ , then  $\psi(y) = \psi(x) = x = y$  (since  $x \in A_\psi$ ). If  $y = m$ , then  $\psi(m) = m$ . That is,  $\psi(y) = y$ . Therefore  $\psi \leq \phi_x$  for all  $x \in A_\psi$ . Hence  $\psi \leq \bigwedge_{x \in A_\psi} \phi_x$ . Let  $y \in L$  such that  $\psi(y) = y$ . If  $y \neq m$ , then  $y \in A_\psi$ .

Now,  $(\bigwedge_{x \in A_\psi} \phi_x)(y) \leq \phi_y(y) = y$ . Hence  $(\bigwedge_{x \in A_\psi} \phi_x)(y) = (\bigwedge_{x \in A_\psi} \phi_x)(y) \wedge y = y$ . Thus  $\bigwedge_{x \in A_\psi} \phi_x \leq \psi$ . Therefore  $\psi = \bigwedge_{x \in A_\psi} \phi_x$ . Let  $B = \{\phi \in$

$\Phi(L) \mid \phi$  is a dual atom and  $\psi \leq \phi$ . Let  $C = \{\phi_x \mid x \in A_\psi\}$ . Let  $x \in A_\psi$ . Then  $\psi \leq \phi_x$  and  $\phi_x$  is dual atom. Therefore  $\phi_x \in B$  and hence  $C \subseteq B$ . Thus  $\psi \leq \bigwedge_{\phi \in B} \phi \leq \bigwedge_{x \in A_\psi} \phi_x = \psi$ . Therefore  $\psi = \bigwedge_{\phi \in B} \phi$ . Thus  $\psi$  is the infimum of set of all dual atoms about it. Hence  $(\Phi(L), \leq)$  is dual atomistic.  $\square$

We note that, for any closure operator  $\phi$  of a complete almost distributive lattice  $L$  and  $x \in L$ ,  $\phi(x) = \phi(m \wedge x) = \phi(x \wedge m)$  by condition (4) of Definition 2.6. Now, we prove the following Lemma.

LEMMA 3.9. *Let  $L$  be a complete ADL with a maximal element  $m$  and  $a \in L$  such that  $a \wedge m \neq m$ . If  $\phi_a$  is a complemented element of  $\Phi(L)$  and if  $\phi'_a$  is the complement of  $\phi_a$ , then  $a \wedge m < \phi'_a(a)$ .*

PROOF: Since  $\phi'_a \in \Phi(L)$ , we get that  $a \wedge m \leq \phi'_a(a)$ . Suppose  $\phi'_a(a) = a \wedge m$ . Also, we have  $\phi_a(a) = a \wedge m$ . Then, by Theorem 2.11(1), we get that  $m = \omega(a) = (\phi_a \vee \phi'_a)(a) = a \wedge m$ , which is a contradiction. Therefore  $a \wedge m < \phi'_a(a)$   $\square$

DEFINITION 3.10. Let  $L$  be a complete ADL with a maximal element  $m$ . Let  $a \in L$  such that  $a \wedge m \neq m$ . Then  $a \wedge m$  is said to be meet-irreducible, if  $a \wedge m = b \wedge c \wedge m$ , then either  $a \wedge m = b \wedge m$  or  $a \wedge m = c \wedge m$ .

DEFINITION 3.11. Let  $L$  be a complete ADL with a maximal element  $m$  and  $x \in L$  such that  $x \wedge m \neq m$ . Then  $x \wedge m$  is said to be completely meet-irreducible, if  $x \wedge m = \bigwedge_{\alpha \in J} (x_\alpha \wedge m)$ , where  $\{x_\alpha \mid \alpha \in J\} \subseteq L$ , then  $x \wedge m = x_\alpha \wedge m$  for some  $\alpha \in J$ .

Now, we prove the following Theorem

THEOREM 3.12. *Let  $L$  be a complete ADL with at least two elements and let  $a \in L$  such that  $a \wedge m \neq m$ . Then  $\phi_a$  is complemented element of  $\Phi(L)$  if and only if  $a \wedge m$  is completely meet-irreducible element of  $L$ .*

PROOF: Let  $a \in L$  such that  $a \wedge m \neq m$ . Suppose  $\phi_a \in \Phi(L)$  is complemented element of  $\Phi(L)$  and suppose  $\phi'_a$  is the complement of  $\phi_a$ . Let  $\{x_\alpha \mid \alpha \in J\} \subseteq L$  such that  $a \wedge m = \bigwedge_{\alpha \in J} (x_\alpha \wedge m)$ . We prove that



$a \wedge m = x_\alpha \wedge m$  for some  $\alpha \in J$ . We have  $a \wedge m \leq x_\alpha \wedge m$  for all  $\alpha \in J$ . Suppose  $a \wedge m < x_\alpha \wedge m$  for all  $\alpha \in J$ . Then  $a \wedge m \neq m$ . Hence, by Lemma 3.9, we get that  $a \wedge m < \phi'_a(a)$ . Now,  $x_\alpha \wedge m = t(x_\alpha \wedge m) = t(x_\alpha) = (\phi_a \wedge \phi'_a)(x_\alpha) = \phi_a(x_\alpha) \wedge \phi'_a(x_\alpha)$ . Since  $a \wedge x_\alpha \wedge m = a \wedge m < x_\alpha \wedge m$ , we get that  $\phi_a(x_\alpha) = m$ . Hence,  $x_\alpha \wedge m = m \wedge \phi'_a(x_\alpha) = \phi'_a(x_\alpha)$ . Now,  $\phi'_a(a) = \phi'_a(a \wedge m) = \phi'_a(\bigwedge_{\alpha \in J} (x_\alpha \wedge m)) \leq \bigwedge_{\alpha \in J} \phi'_a(x_\alpha \wedge m) = \bigwedge_{\alpha \in J} (x_\alpha \wedge m) = a \wedge m$ . Hence  $\phi'_a(a) = \phi'_a(a) \wedge a \wedge m = a \wedge m$ , which is a contradiction.

Therefore, there exists  $\alpha \in J$  such that  $a \wedge m = x_\alpha \wedge m$ . Thus  $a \wedge m$  is completely meet-irreducible. Conversely, assume that  $a \wedge m$  is completely meet-irreducible. Let  $B = \{b \in L \mid b \wedge m \neq a \wedge m\}$ . Since  $m \neq a \wedge m$ , we get that  $m \in B$ . Hence  $B \neq \emptyset$ . Let  $\psi = \bigwedge_{b \in B} \phi_b$ . Now, we prove that  $\psi$  is a complement of  $\phi_a$  in the lattice  $(\Phi(L), \vee, \wedge)$ .

Let  $x \in L$ . If  $x \wedge m = a \wedge m$ , then  $(\phi_a \wedge \psi)(x) = \phi_a(x) \wedge \psi(x) = x \wedge m \wedge \psi(x) = x \wedge \psi(x) = x \wedge \psi(x) \wedge m$  (since  $\psi(x) \leq m$ )  $= \psi(x) \wedge x \wedge m = x \wedge m = t(x)$ . If  $x \wedge m \neq a \wedge m$ , then  $x \in B$  and hence  $\psi \leq \phi_x$ . Now,  $(\phi_a \wedge \psi)(x) = \phi_a(x) \wedge \psi(x) \leq \psi(x) \leq \phi_x(x) = x \wedge m$ . Hence  $(\phi_a \wedge \psi)(x) = (\phi_a \wedge \psi)(x) \wedge x \wedge m = x \wedge m$ . Thus  $(\phi_a \wedge \psi)(x) = x \wedge m = t(x)$  for all  $x \in L$ . Therefore  $\phi_a \wedge \psi = t$ . Now, we prove that  $\phi_a \vee \psi = \omega$ . Let  $b \in B$ . Then  $a \wedge m \neq b \wedge m$  and  $a \wedge m \neq m$ . So that  $\phi_b(a) \neq a \wedge m$  for all  $b \in B$ . Since  $a \wedge m$  is completely meet-irreducible, we get that  $(\bigwedge_{b \in B} \phi_b)(a) \neq a \wedge m$ . Thus  $\psi(a) \neq a \wedge m$ . Let  $x \in F_{\phi_a} \cap F_\psi$ .

Then  $\phi_a(x) = x$  and hence  $x = a \wedge m$  or  $x = m$ . Suppose  $x = a \wedge m$ . We have  $x \in F_\psi$ . So that  $\psi(x) = x$  and hence  $\psi(a \wedge m) = a \wedge m$ . Therefore  $\psi(a) = a \wedge m$ . Which is not true. Hence  $x = m$ . Thus  $F_{\phi_a} \cap F_\psi = \{m\}$ . So that  $F_{\phi_a} \cap F_\psi = F_\omega$  (since  $F_\omega = \{x \in L \mid \omega(x) = x\} = \{m\}$ ). Hence, by Lemma 3.5,  $F_{\phi_a \vee \psi} = F_\omega$  and, by Lemma 3.4,  $\phi_a \vee \psi = \omega$ . Therefore  $\psi$  is the complement of  $\phi_a$  in the lattice  $(\Phi(L), \vee, \wedge)$ .  $\square$

**THEOREM 3.13.** *If  $L$  is a complete ADL with at least two elements and  $a \in L$  such that  $a \wedge m \neq m$ , then  $\phi_a \in \Phi(L)$  is complemented if and only if  $a \wedge m < \bigwedge_{b \in B} \phi_b(a)$ , where  $B = \{b \in L \mid b \wedge m \neq a \wedge m\}$ .*

**PROOF:** Let  $a \in L$  such that  $a \wedge m \neq m$ . Then, by Theorem 2.11(2),  $\phi_a$  is a dual atom of  $L$ . Suppose  $\phi_a$  is complemented. Let  $\psi = \bigwedge_{b \in B} (\phi_b)$ . Then from the proof of the Theorem 3.12, we get that  $\psi$  is a complement of  $\phi_a$  and  $\psi(a) > a \wedge m$ .

Hence  $(\bigwedge_{b \in B} \phi_b)(a) = \psi(a) > a \wedge m$ . Thus  $\bigwedge_{b \in B} (\phi_b(a)) > a \wedge m$ . Conversely, assume the condition. Now, we prove that  $\phi_a$  is complemented. It is enough to prove that  $a \wedge m$  is completely meet-irreducible element in  $L$ . Let  $\{\alpha_\alpha \mid \alpha \in J\} \subseteq L$  such that  $a \wedge m = \bigwedge_{\alpha \in J} (x_\alpha \wedge m)$ . We prove that there exists  $\alpha \in J$  such that  $a \wedge m = x_\alpha \wedge m$ . Suppose  $a \wedge m \neq x_\alpha \wedge m$  for all  $\alpha \in J$  and hence  $a \wedge m < x_\alpha \wedge m$  for all  $\alpha \in J$ . Therefore  $\{x_\alpha \mid \alpha \in J\} \subseteq B$ . Now,  $a \wedge m < \bigwedge_{b \in B} (\phi_b(a)) = (\bigwedge_{b \in B} \phi_b)(a) \leq (\bigwedge_{\alpha \in J} \phi_{x_\alpha})(a) = \bigwedge_{\alpha \in J} (\phi_{x_\alpha}(a)) = \bigwedge_{\alpha \in J} (x_\alpha \wedge m) = a \wedge m$  (since  $x_\alpha \wedge m \wedge a \wedge m = a \wedge m$ ). Thus  $a \wedge m < \bigwedge_{\alpha \in J} (x_\alpha \wedge m) = a \wedge m$ , which is a contradiction. Therefore there exists  $\alpha \in J$  such that  $a \wedge m = x_\alpha \wedge m$ . Hence  $a \wedge m$  is completely meet-irreducible. Therefore  $\phi_a$  is complemented.  $\square$

**THEOREM 3.14.** *If  $\phi_0$  is a dual atom of  $\Phi(L)$ , then there is at most one complement of  $\phi_0$ .*

**PROOF:** Suppose  $0$  is not completely meet-irreducible, then by Theorem 3.12, we get that  $\phi_0$  is not complemented. Suppose  $0$  is completely meet-irreducible, then by the proof of the Theorem 3.12,  $\phi_0$  is complemented. Also  $\psi = \bigwedge_{b \in B} \phi_b$ , where  $B = \{b \in L \mid b \wedge m \neq 0\}$  is the complement of  $\phi_0$  and  $\psi(0) \neq 0$ . Now, we prove that  $\phi_0$  has at most one complement. Suppose  $\phi'_0$  is another complement of  $\phi_0$ . Let  $x \in L - \{0\}$ . If  $x \notin B$ , then  $x \wedge m = 0$  and hence  $x = x \wedge m \wedge x = 0 \wedge x = 0$ , which is a contradiction. Therefore  $x \in B$ . We have  $\psi = \bigwedge_{b \in B} \phi_b$ , so that  $\psi(x) = (\bigwedge_{b \in B} \phi_b)(x) \leq \phi_x(x) = x \wedge m$ . Hence  $\psi(x) = \psi(x) \wedge x \wedge m = x \wedge m$ , by condition (2) of Definition 2.6. We have  $x \wedge m \leq \phi'_0(x)$  for all  $x \in L$ . If  $x = 0$ , then  $0 = 0 \wedge m < \phi'_0(0)$ . If  $x \neq 0$ , then  $x \wedge m \leq \phi'_0(x)$ . Therefore  $\phi'_0(x) \neq 0$  for all  $x \in L$  and hence  $\phi'_0(x) \in L - \{0\} = B$ . Hence  $\psi(\phi'_0(x)) = (\bigwedge_{b \in B} \phi_b)(\phi'_0(x)) \leq \phi_{\phi'_0(x)}(\phi'_0(x)) = \phi'_0(x) \wedge m = \phi'_0(x)$ , since  $\phi'_0(x) \wedge \phi'_0(x) = \phi'_0(x)$ . Since  $x \wedge m \leq \phi'_0(x)$  for all  $x \in L$ , we get that  $\psi(x) \leq \psi(\phi'_0(x)) \leq \phi'_0(x)$  for all  $x \in L$ . Hence  $\psi \leq \phi'_0$ . Let  $a \in L - \{0\}$ . Then  $a \wedge m = t(a) = (\phi_0 \wedge \phi'_0)(a) = \phi_0(a) \wedge \phi'_0(a) = m \wedge \phi'_0(a)$ , since  $0 \wedge a \neq a$ , so that  $\phi_0(a) = m = \phi'_0(a)$ . Suppose  $\phi_a(x) = x$  for  $x \in L - \{0\} = B$ . Case (i)  $a \wedge x = x$ . Then  $x = a \wedge m$ . Hence  $\phi'_0(x) = \phi'_0(a \wedge m) = \phi'_0(a) = a \wedge m = x$ . Case (ii)

$a \wedge x \neq x$ . Then  $x = m$ . Hence  $\phi'_0(x) = \phi'_0(m) = m = x$ . Thus we have proved that  $\phi_a(x) = x \Rightarrow \phi'_0(x) = x$  for all  $x \in B$ . By Theorem 2.11(4), we get that  $\phi'_0 \leq \phi_x$  for all  $x \in B$ . Hence  $\phi'_0 \leq \bigwedge_{x \in B} \phi_x = \psi$ . Therefore  $\phi'_0 \leq \psi$ . Thus  $\phi'_0 = \psi$ .  $\square$

LEMMA 3.15. *Let  $L$  be a complete almost distributive lattice. Then, for any  $\phi, \psi \in \Phi(L)$  and  $x \in L$  we have*

- (i)  $(\phi \vee \psi)(x) \geq \psi(\phi(x))$
- (ii)  $(\phi \vee \psi)(x) \geq \psi(\phi(\psi(x)))$ .

PROOF: Let  $\phi, \psi \in \Phi(L)$  and  $x \in L$ . Then  $(\phi \vee \psi)(x) \geq \psi(x)$  and hence  $(\phi \vee \psi)((\phi \vee \psi)(x)) \geq (\phi \vee \psi)(\psi(x)) \geq \psi(\phi(x))$ . Therefore  $(\phi \vee \psi)(x) \geq \psi(\phi(x))$ . Since  $(\phi \vee \psi)(x) \geq \phi(\psi(x))$  and hence  $(\phi \vee \psi)(\phi \vee \psi)(x) \geq (\phi \vee \psi)(\phi(\psi(x))) \geq \psi(\phi(\psi(x)))$ .  $\square$

COROLLARY 3.16. *Let  $L$  be a complete almost distributive lattice. Then, for any  $\phi, \psi \in \Phi(L)$  and  $x \in L$  we have*

- (i)  $(\phi \vee \psi)(x) \geq \phi(\psi(x))$
- (ii)  $(\phi \vee \psi)(x) \geq \phi(\psi(\phi(x)))$

Finally, we conclude with the following Theorem.

THEOREM 3.17. *If  $\phi_a \in \Phi(L)$  is complemented, then there is at least one complement of  $\phi_a$  preceding  $\phi_0$ .*

PROOF: Let  $0 \neq a \wedge m \neq m$ . Let  $\phi'_a$  be a complement of  $\phi_a$ . Let  $x \in L$ . Then  $(\phi_a \wedge (\phi'_a \wedge \phi_0))(x) = (\phi'_a \wedge \phi_a \wedge \phi_0)(x) = (t \wedge \phi_0)(x) = t(x)$ . Hence  $(\phi_a \wedge (\phi'_a \wedge \phi_0)) = t$ . Now, we prove that  $(\phi_a \vee (\phi'_a \wedge \phi_0)) = \omega$ . Case (i)  $a \wedge x = x$ . Then  $\phi_a(x) = a \wedge m$ . By above Lemma 3.15, we get that  $(\phi_a \vee (\phi'_a \wedge \phi_0))(x) \geq \phi_a((\phi'_a \wedge \phi_0)(\phi_a(x))) = \phi_a((\phi'_a \wedge \phi_0)(a \wedge m)) = \phi_a(\phi'_a(a) \wedge \phi_0(a)) = \phi_a(\phi'_a(a) \wedge m) = \phi_a(\phi'_a(a)) = m = \omega(x)$ . Therefore  $(\phi_a \vee (\phi'_a \wedge \phi_0)) = \omega$ . Case (ii) if  $a \wedge x \neq x$ , then  $\phi_a(x) = m$ . Again, by above Lemma 3.15, we get that  $(\phi_a \vee (\phi'_a \wedge \phi_0))(x) \geq (\phi'_a \wedge \phi_0)(\phi_a(x)) = \phi'_a(m) \wedge \phi_0(m) = m \wedge m = m = \omega(x)$ . Therefore  $(\phi_a \vee (\phi'_a \wedge \phi_0)) = \omega$ . Thus  $(\phi_a \vee (\phi'_a \wedge \phi_0)) = \omega$ .  $\square$

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