**B**ulletin of the **S**ection of **L**ogic
Volume 44:1/2 (2015), pp. 15–17
https://doi.org/10.18778/0138-0680.44.1.2.02

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# A NOTE ON SOME CHARACTERIZATION OF DISTRIBUTIVE LATTICES OF FINITE LENGTH

**Abstract**

Using known facts we give a simple characterization of the distributivity of lattices of finite length.

*Keywords:* distributive lattice, finite length.

All lattice-theoretic notions that we use in the paper are standard and one can find them in [3]. Let $L$ be an arbitrary lattice. Consider the following conditions on $L$ ($\mathbb{B}$ stands for the class of all Boolean lattices):

**($A1$)** $(\forall x, y, z \in L)([x, y], [x, z] \in \mathbb{B} \Rightarrow [x, y \vee z] \in \mathbb{B})$,
**($A2$)** $(\forall x, y, z \in L)([x, z], [y, z] \in \mathbb{B} \Rightarrow [x \wedge y, z] \in \mathbb{B})$.

First of all, let us observe that ($A1$) and ($A2$) are equivalent on the ground of modularity. Indeed, assume ($A2$) and fix Boolean intervals $[x, y]$, $[x, z]$. Since $y \wedge z \in [x, y]$, let $a$ be its complement in $[x, y]$. Then obviously $\{x, a, z, y \vee z\}$ forms a sublattice of $L$, so by the Isomorphism Theorem (see [3], Theorem IV.1.2) $[a, y \vee z]$ is a Boolean interval. Similarly, $y \wedge z \in [x, z]$, so let $b$ be its complement in $[x, z]$. Then $\{x, y, b, y \vee z\}$ forms a sublattice of $L$, so once again by the Isomorphism Theorem $[b, y \vee z]$ is a Boolean interval. Now we can use assumption ($A2$) to obtain $[a \wedge b, y \vee z] \in \mathbb{B}$, so $[x, y \vee z] \in \mathbb{B}$.

Special cases of ($A1$) and ($A2$) are the following (symbol $\mathbf{4}$ denotes a four-element Boolean lattice):

**($B1$)** $(\forall x, y \in L)(x \wedge y \prec x, y \Rightarrow [x \wedge y, x \vee y] \cong \mathbf{4})$,
**($B2$)** $(\forall x, y \in L)(x, y \prec x \vee y \Rightarrow [x \wedge y, x \vee y] \cong \mathbf{4})$.

Theorem. *Let $L$ be a lattice of finite length. The following conditions are equivalent:*

1. *$L$ is distributive.*
2. *$(A1)$ and $(A2)$.*
3. *$(B1)$ and $(B2)$.*

Proof. (1) $\Rightarrow$ (2). Let us assume that $L$ is a distributive lattice; we will prove condition $(A1)$. Assume that $[x, y]$, $[x, z]$ are Boolean intervals. Obviously $[x, y \vee z]$ is distributive, so it sufficient to show that this interval is complemented. Indeed, if $a \in [x, y \vee z]$, then let $b$ stand for the complement of $a \wedge y$ in $[x, y]$, and $c$ stand for the complement of $a \wedge z$ in $[x, z]$. Hence, by the distributivity of $[x, y \vee z]$, one can easily verify that $b \vee c$ is a complement of $a$ in $[x, y \vee z]$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). It it clear that $(B1)$ implies that $L$ satisfies the Birkhoff's condition, i.e.

$$(\forall x, y \in L)(x \wedge y \prec x, y \Rightarrow x, y \prec x \vee y),$$

so by Theorem II.16 in [1] both conditions $(B1)$ and $(B2)$ imply the modularity of $L$. Suppose that $L$ is not distributive. Then there is a diamond $D = \{o, a, b, c, i\}$ contained in $L$ such that $o \prec a, b, c \prec i$ (see [2], p. 270). Hence, by $(B1)$ interval $[o, i]$ is a four-element Boolean lattice which is a contradiction. Finally, $L$ is distributive. ■

Remark 1. The complementation of $[x, y \vee z]$ in the proof of (1) $\Rightarrow$ (2) can be obtained without the assumption of distributivity. Applying Theorem IV.6 in [1] one can prove that for a lattice of a finite length satisfying the Birkhoff's condition holds: if intervals $[x, y]$, $[x, z]$ are complemented, then interval $[x, y \vee z]$ is also complemented.

Remark 2. The theorem characterizes distributivity in a simple language of two and four-element Boolean intervals. Hence, one can say that it gives a local method (i.e. a method focused on intervals of length 2 only) to establish the global notion—distributivity. From this point of view, it corresponds to the characterization obtained on the ground of theory of tolerance relations (a finite distributive lattice can be regarded as a gluing of its maximal Boolean intervals, see [4]) and to the characterization by means of so called Wroński's sum (the class of all distributive lattices coincides with the closure of the class of all finite Boolean lattices with respect to Wroński's sum, see [5]). Moreover, it guarantees that the following natural and simple algorithm correctly decides whether a lattice is distributive or not: to verify $(B1)$ take an element $x$, find all of its successors and for distinct successors $y$ and $z$ check whether interval $[x, y \vee z]$ has four elements; and similarly verify $(B2)$.

Acknowledgement. We are grateful to anonymous referees for their important and valuable remarks.

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