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POSITIVE COMPLETE THEORIES AND POSITIVE STRONG AMALGAMATION PROPERTY

Abstract

We introduce the notion of positive strong amalgamation property and we investigate some universal forms and properties of this notion.
Considering the close relationship between the amalgamation property and the notion of complete theories, we explore the fundamental properties of positively complete theories, and we illustrate the behaviour of this notion by bringing changes to the language of the theory through the groups theory.

Keywords: h-inductive theory, existentially closed, complete theory, positive amalgamation, positive strong amalgamation, mathematical model, semantics.

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1. Positive complete theories

1.1. Positive logic

The positive model theory in its present form was introduced by Ben Yaccov and Poizat [5] following the line of research of Hrushovski [3] and Pillay [4]. It is considered as a part of the eastern model theory introduced by Abraham Robinson, wich is concerned essentially with the study of existentially closed models and model-complete theories in the context of incomplete inductive theories. The main tools in the study of incomplete inductive theories are embedding, existential formulas and inductive sentences. Keep in consideration homomorphisms and positive formulas,

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the positive logic offers a wider and simpler framework as compared to the eastern model theory.

In this subsection we summarize the basic concepts of positive logic which will be used throughout the paper.

Let L be a first order language. we stipulate that L includes the symbol of equality and the constant $\perp$ denoting the antilogy.
The quantifier-free positive formulas are built from atomics by using the connectives $\wedge$ and $\vee$. The positive formulas are of the form: $\exists \bar{x} \varphi(\bar{x}, \bar{y})$, where φ is quantifier-free positive formula. The variables $\bar{x}$ in the expression of the φ are said to be free.
The simple h-inductive sentences are the formulas without free variables that can be written in the form:

$$\forall \bar{x}(\exists \bar{y} \varphi(\bar{x}, \bar{y}) \rightarrow \exists \bar{z} \psi(\bar{x}, \bar{z})).$$

where φ and ψ are quantifier-free positive formulas.
A sentence is said to be h-inductive if it is a finite conjunction of simple h-inductive sentences.
The h-universal sentences are the sentences that can be written as negation of a positive sentence. Note that the conjunction (resp, disjunction) of two h-universal sentences is equivalent to an h-universal sentence.

Let A and B be two L-structures and f a mapping from A into B. f is said to be

- a homomorphism, if for every tuple $\bar{a}$ from A and for every atomic formula ϕ, $A \models \phi(\bar{a})$ implies $B \models \phi(f(\bar{a}))$. In this case we say that B is a continuation of A.

- an embedding, if f is a homomorphism such that for every atomic formula ϕ; $A \models \phi(\bar{a})$ if and only if $B \models \phi(f(\bar{a}))$.

- an immersion whenever $\bar{a} \in A$ and $f(\bar{a})$ satisfy the same L-positive formulas, for every $\bar{a} \in A$.

For every L-structure A, we denote by $L(A)$ the language obtained from L by adjoining the element of A as constants. Let $Diag(A)$ (resp. $Diag^{+}(A)$) the set of atomic and negated atomic (resp. positive quantifier-free) sentences satisfied by A in the language $L(A)$.
We denote by $Diag^{+\star}(A)$ the set of L-sentences $\exists \bar{x}\ \varphi(\bar{x})$ satisfied by A where $\varphi(\bar{x})$ is a quantifier-free positive formula.

Definition 1.1. A model M of an h-inductive theory T is said to be positively closed (in short; pc) if every homomorphism from M to a model of T is an immersion.

A class of L-structures is said to be h-inductive if it is closed with respect to the inductive limit of homomorphisms. For more details on the notion of h-inductive sequences and limits, the reader is invited to [5].
In [5, Théorème 1, lemme 12] it is shown that every member of an h-inductive class is continued in a pc member of the class, and the pc models of an h-inductive theory form an h-inductive class.

1.2. Positive complete and T-complete theories

Definition 1.2. Two h-inductive L-theories are said to be companion if they have the same pc models.

Every h-inductive theory T has a maximal companion denoted $T_k(T)$, called the Kaiser's hull of T which is the h-inductive theory of the pc models of T. Likewise, T has a minimal companion denoted $T_u(T)$, formed by its h-universal consequences sentences.

Note that if T' is an h-inductive theory such that $T_u(T) \subseteq T' \subseteq T_k(T)$ then T' and T are companion theories.

Definition 1.3. Let T be an h-inductive theory.

- T is said to be model-complete if every model of T is a pc model of T.
- We say that T has a model-companion whenever $T_k(T)$ is model-complete.

Let A be a L-structure and B a subset of A. We shall use the following notations:

- $T_i(A)$ (resp. $T_u(A)$) denote the set of h-inductive (resp. h-universal) $L(A)$-sentences satisfied by A.
- $T_i^\star(A)$ (resp. $T_u^\star(A)$) denote the set of h-inductive (resp. h-universal) L-sentences satisfied by A.
- $T_k(A)$ (resp. $T_k^\star(A)$) denote the Kaiser's hull of $T_i(A)$ (resp. of $T_i^\star(A)$).

- $T_i(A|B)$ (resp. $T_u(A|B)$) denote the set of h-inductive (resp. h-universal) $L(B)$-sentences satisfied by A.

Definition 1.4. Let A and B two L-structures and f a homomorphism from A into B. f is said to be a strong immersion if B is a model of $T_i(A)$ in the language $L(A)$.

Definition 1.5.

- An h-inductive theory T is said to be positively complete or it has the joint continuation (in short JC) property if any two models of T have a common continuation in a model of T.
- Let T_1, T_2 and T three h-inductive L-theories. T_1 and T_2 are said to be T-complete if for every models A of T_1 and B of T_2, there is C a common continuation of A and B such that $C \vdash T$.

The following remark lists some simple properties which will be useful in the rest of the paper.

Remark 1.6. Let A and B two L-structures and T an h-inductive L-theory.

1. $T_u(A) \cup Diag^{+}(A) \subseteq T_i(A)$ and $T_u^{\star}(A) \cup Diag^{+\star}(A) \subseteq T_i^{\star}(A)$.
2. A is a pc model of $T_i(A)$, and $T_i(A) = T_k(A)$.
3. $T_u(T)$ (resp. $T_u(A)$) is the h-universal part of $T_k(T)$ (resp. of $T_i(A)$). The same is true for $T_u^{\star}(A)$ and $T_i^{\star}(A)$.
4. $T_i(A) \subseteq T_i(B) \Rightarrow T_u(A) \subseteq T_u(B)$.
5. $T_i^{\star}(A) \subseteq T_i^{\star}(B) \Rightarrow T_u^{\star}(A) \subseteq T_u^{\star}(B)$.
6. If T is has the JC property, then for every pc model A of T we have; $T_k(T) = T_i^{\star}(A)$ and $T_u(T) = T_u^{\star}(A)$.
7. If A and B are pc models of T and B is a continuation of A then $T_i^{\star}(A) = T_i^{\star}(B)$.
8. If A is continued in B then $T_u^{\star}(B) \subseteq T_u^{\star}(A)$, and $T_u(B|A) \subseteq T_u(A)$ in the language $L(A)$.
9. If A is immersed in B then $T_u^{\star}(A) = T_u^{\star}(B)$, $T_i^{\star}(B) \subseteq T_i^{\star}(A)$ and $T_u(B|A) = T_u(A)$ in the language $L(A)$.
10. $T_u^{\star}(A) = \{\neg\exists\bar{x}\varphi(\bar{x}) \;\mid \exists\bar{x}\varphi(\bar{x}) \notin Diag^{+\star}(A)\}$.

11. $Diag^{+\star}(A) \subseteq Diag^{+\star}(B) \Leftrightarrow T_u^\star(B) \subseteq T_u^\star(A)$.

12. If $T_u^\star(A) \subseteq T_u^\star(B)$ (resp. $T_i^\star(A) \subseteq T_i^\star(B)$), then $T \cup Diag^+(A) \cup Diag^+(B)$ is consistent in the language $L(A \cup B)$. Indeed, if the set $T \cup Diag^+(A) \cup Diag^+(B)$ is inconsistent, then there are $\varphi(\bar{a}) \in Diag^+(A)$ and $\psi(\bar{b}) \in Diag^+(B)$ such that $T \vdash \neg\exists\bar{x}, \bar{y}(\varphi(\bar{x}) \wedge \psi(\bar{y}))$. So $\neg\exists\bar{x}, \bar{y}(\varphi(\bar{x}) \wedge \psi(\bar{y})) \in T_u^\star(A)$. Given that $A \models \varphi(\bar{a})$ then $\neg\exists\bar{y}\psi(\bar{y}) \in T_u^\star(A)$. By hypothesis $\neg\exists\bar{y}\psi(\bar{y}) \in T_u^\star(B)$, which contradicts the fact that $B \models \psi(\bar{b})$.

13. If $T_u^\star(A) \subseteq T_u^\star(B)$ then $Diag^+(A) \cup Diag^+(B)$ is consistent in the language $L(A \cup B)$.

14. For every pc models A and B of T, if $T_u^\star(A) = T_u^\star(B)$ then $T_i^\star(A) = T_i^\star(B)$.

15. T_1 and T_2 are T-complete if and only if for every $A \vdash T_1$ and $B \vdash T_2$, $Diag^+(A) \cup Diag^+(B) \cup T$ is $L(A \cup B)$-consistent.

16. Let (T_1, T_2) be a pair of T-complete theories. For every T_1', T_2' and T' companion theories of T_1, T_2 and T respectively, the pair (T_1', T_2') is T'-complete.

LEMMA 1.7. *Let A be a pc model of an h-inductive L-theory T, then*

1. *$T_u^\star(A)$ is minimal in the set $\{T_u^\star(B) \mid B \models T\}$.*

2. *$T_i^\star(A)$ is maximal in the set $\{T_i^\star(B) \mid B \models T\}$.*

PROOF:

1. Let B a model of T such that $T_u^\star(B) \subseteq T_u^\star(A)$. By the property 12 of the Remark 1.6, there exists C a model of T that is a common continuation of A and B. Given that A is a pc model, by the properties 8 and 9 of the Remark 1.6 we obtain:

$$T_u^\star(A) = T_u^\star(C) \subseteq T_u^\star(B).$$

2. Let B a model of T such that $T_i^\star(A) \subseteq T_i^\star(B)$. We claim that $Diag^+(A) \cup T_i(B)$ is consistent in the language $L(A \cup B)$. Indeed, if not, by compactness there exists $\psi(\bar{a}) \in Diag^+(A)$ such that

$T_i(B) \models \neg\exists\bar{x}\psi(\bar{x})$. Given that $T_i^\star(B)$ is the part of $T_i(B)$ without parameters of B, then $T_i^\star(B) \models \neg\exists\bar{x}\psi(\bar{x})$. On the other hand since

$$\exists\bar{x}\psi(\bar{x}) \in Diag^{+\star}(A) \subset T_i^\star(A) \subseteq T_i^\star(B),$$

a contradiction. Thereby $Diag^+(A) \cup T_i(B)$ is consistent in the language $L(A \cup B)$, which implies the existence of a model D of $T_i(B)$ in the language $L(A \cup B)$, such that

$$A \xrightarrow{f} D \xleftarrow{g} B.$$

where f is an homomorphism and g an immersion.
Given that D is also a model of T and A pc model of T, then f is an immersion. By the property 9 of the Remark 1.6 we obtain

$$T_i^\star(B) \subseteq T_i^\star(D) \subseteq T_i^\star(A) \subseteq T_i^\star(B). \qquad \square$$

LEMMA 1.8. *Let T_1, T_2 and T three h-inductive L-theories. T_1 and T_2 are T-complete if and only if one of the following holds:*

1. *For every free-quantifier positive formulas $\varphi(\bar{x})$, If $T \vdash \neg\exists\bar{x}\varphi(\bar{x})$ then $T_1 \vdash \neg\exists\bar{x}\varphi(\bar{x})$ and $T_1 \vdash \neg\exists\bar{x}\varphi(\bar{x})$.*
2. $T_u(T) \subseteq T_u(T_1) \cap T_u(T_2)$.

PROOF:

1. Suppose that T_1, T_2 and T satisfy the property 1 of the Lemma. Let A and B models of T_1 and T_2 respectively. We claim that $Diag^+(A) \cup Diag^+(B) \cup T$ is $L(A \cup B)$-consistent. If not, there are $\varphi(\bar{a}) \in Diag^+(A)$ and $\psi(\bar{b}) \in Diag^+(B)$ such that $T \vdash \neg(\exists\bar{x}\varphi(\bar{x}) \wedge \exists\bar{y}\psi(\bar{y}))$. Thereby $T_1 \vdash \neg(\exists\bar{x}\varphi(\bar{x}) \wedge \exists\bar{y}\psi(\bar{y}))$ and $T_2 \vdash \neg(\exists\bar{x}\varphi(\bar{x}) \wedge \exists\bar{y}\psi(\bar{y}))$, a contradiction.

2. Suppose that T_1 and T_2 are T-complete. Since every model of T_1 or T_2 can be continued in a model of T then $T_u(T) \subseteq T_u(T_1) \cap T_u(T_2)$.

3. It is clear that if $T_u(T) \subseteq T_u(T_1) \cap T_u(T_2)$ then T, T_1 and T_2 satisfy the property 1. $\square$

LEMMA 1.9. *An h-inductive T theory has the JC property if and only if it satisfies one of the following properties:*

1. *For any free-quantifier positive formulas $\varphi(\bar{x})$ and $\psi(\bar{y})$, if $T \vdash \neg\exists\bar{x}\varphi(\bar{x}) \vee \neg\exists\bar{y}\psi(\bar{y})$ then $T \vdash \neg\exists\bar{x}\varphi(\bar{x})$ or $T \vdash \neg\exists\bar{y}\psi(\bar{y})$.*
2. *$T_u(T) = T_u^\star(A)$ for some model A of T.*
3. *$T_k(T) = T_i^\star(A)$ for some model A of T.*
4. *For every pc models A and B of T we have $T_u^\star(A) = T_u^\star(B)$.*

PROOF:

1. Clear
2. Let T be an h-inductive theory and A a model of T such that $T_u(T) = T_u^\star(A)$. Let B and C two pc models of T. Given that $T_u(T) = T_u^\star(A) \subseteq T_u^\star(B) \cap T_u^\star(C)$, by the minimality of the h-universal theory of the pc models (Lemma 1.7), we obtain
$$T_u^\star(A) = T_u^\star(B) = T_u^\star(C).$$
From the property 13 of the Remark 1.6, it follows that there is a common continuation of B and C by a model of T. Thereby T has the JC property.
The other direction follows from the property 6 of the Remark 1.6.
3. Let A be a model of T such that $T_k(T) = T_i^\star(A)$. Let B and C be two pc models of T. Since
$$T_i^\star(A) = T_k(T) \subseteq T_i^\star(B) \cap T_i^\star(C)$$
then $T_u^\star(A) \subseteq T_u^\star(B) \cap T_u^\star(C)$. By Lemma 1.7 we obtain
$$T_u^\star(A) = T_u^\star(B) = T_u^\star(C).$$
By the property 13 of the Remark 1.6, we get a common continuation of B and C by a model of T. Thereby T has the JC property.
The other direction follows from the property 6 of the Remark 1.6.
4. Clear. □

Lemma 1.10. *Let A be a L-structure. The theories $T_u^\star(A)$ and $T_i^\star(A)$ are companion and positively completes.*

Proof: It is clear that every model of $T_i^\star(A)$ is a model of $T_u^\star(A)$.
Now, we will show that every model of $T_u^\star(A)$ is continued into a model of $T_i^\star(A)$. Let B be a model of $T_u^\star(A)$, we claim that $Diag^+(B) \cup T_i^\star(A)$ is consistent in the language $L(B)$. Indeed, otherwise, there exists $\psi(\bar{b}) \in Diag^+(B)$ such that $T_i^\star(A) \models \neg\exists\bar{x}\psi(\bar{x})$, so $\neg\exists\bar{x}\psi(\bar{x}) \in T_u^\star(A)$. Given that $T_u^\star(A) \subseteq T_u^\star(B)$ and $\exists\bar{x}\psi(\bar{x}) \in Diag^{+\star}(B)$, a contradiction. Thereby $Diag^+(B) \cup T_i^\star(A)$ is consistent, so B is continued in a model of $T_i^\star(A)$.

The second part of the lemma results from the properties 2 and 3 of the lemma 1.9, since $T_u(T_u^\star(A)) = T_u^\star(A)$ and $T_i(T_i^\star(A)) = T_i^\star(A)$. □

Remark 1.11.

- We have the same results of the lemma 1.10 for the theories $T_u(A|B)$ and $T_i(A|B)$, where B is a subset of A.
- Let A_e be a pc model of an h-inductive theory T. Let A be a subset of A_e. Every pc model of $T_u(A_e|A)$ in the language $L(A)$ is a pc model of T in the language L.
  Indeed, Let B_e be a pc model of $T_u(A_e|A)$, since $T_u(A_e|A)$ is positively complete, there is a common continuation C of A_e and B_e in the language $L(A)$ which in turn can be continued in a pc model C_e of T. As A_e is immersed in C_e, so C_e is a model of $T_u(A_e|A)$, then B_e is immersed in C_e, which implies that B_e is a pc model of T.
- Let A and B be two models of an h-inductive theory T. If A is immersed in B then B is continued in a pc model of $T_i(A)$ in the language $L(A)$. Indeed, Since A is immersed in B then B is a model of $T \cup Diag^+(A) \cup T_u(A)$ in the language $L(A)$. let C be a pc model of $T_u(A)$ in which B is continued, then C is a pc model of $T_i(B)$ (first bullet of the Remark 1.11) and B is continued in C.

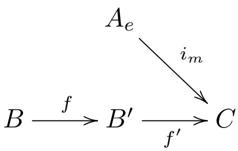
Lemma 1.12. *Let T be a positively complete h-inductive L-theory and A_e a pc model of T that is also a pc model of an h-inductive L-theory T'. Then every pc model of T is a pc model of T', and every pc model of T' that is a model of T is a pc model of T.*

Proof: Given that A_e is a pc model of T', then $T' \subset T_k(T) = T_i^\star(A_e)$. Let B be a pc model of T, since $T_i^\star(A_e) = T_i^\star(B)$ then B is a model of T'.

Let f be a homomorphism from B into B' a pc model of T'. By the property 8 of the Remark 1.6, we have

$$T_u^\star(B') \subseteq T_u^\star(B) = T_u(T) = T_u^\star(A_e).$$

Now by the property 12 of the Remark 1.6, we obtain the consistency of $T' \cup Diag^+(A_e) \cup Diag^+(B')$,then we get the following diagram:



where C is a model of T' that we can suppose a pc model of T'. We deduce the following equalities:

$$T_k(T) = T_i^\star(B) = T_i^\star(A_e) = T_i^\star(C) = T_i^\star(B').$$

Thereby f is an immersion, and B is a pc model of T'.

For the second part of the lemma. Let B_e be a pc model of T' such that $B_e \vdash T$, let f be a homomorphism from B_e into a pc model B of T. Given that B is also a pc model of T', then f is an immersion, and so B_e is a pc model of T. □

Corollary 1.13. Let T be an h-inductive theory and A a pc model of T. Every pc model of the L-theory $T_i^\star(A)$ is a pc model of T, and every pc model of T which is a model of $T_i^\star(A)$ is a pc model of $T_i^\star(A)$.

Proof: The corollary follows directly from the fact that $T_i^\star(A)$ is positively complete and A is a common pc of T and $T_i^\star(A)$. □

Remark 1.14. Let be A_e a pc model of T and $A \subseteq A_e$. Let $< A >$ be the L-substructure of A_e generated by A. Given that $T_u(A_e| < A >)$ and $T_u(A_e|A)$ are positively complete theories and A_e is a common pc model of $T_u(A_e| < A >)$ and $T_u(A_e|A)$, it follows from Lemma 1.12 that $T_u(A_e| < A >)$ and $T_u(A_e|A)$ are companion theories.

The following example list some anomaly situations in the positive logic that we will try to deal by some changes focused on the language and the theories.

Example 1.15.

1. Let T_{pos} the h-inductive theory of posets in the relational language $L = \{\leq\}$. T_{pos} is positively complete and has only one pc model which is the trivial structure $(\{x\}, \leq)$.
2. Let $L = \{f\}$ be the language formed by 1-ary function symbol f.
   (a) For every integer n, let T_n be the h-inductive theory $\{\exists x\ f^n(x) = x\}$. For every n, the theory T_n is positively complete and has only one pc model which is the structure $(\{x\}, f)$ such that $f(x) = x$.
   (b) For every integer n, let T'_n be the h-inductive theory $\{\neg\exists x\ f^n(x) = x\}$. We can consider the models of T'_n as directed graphs such that the vertexes of the graph are the element of the structure, and two vertexes a and b are jointed by an edge pointed from a into b if $f(a) = b$. The theory T'_n is positively complete and has only one pc model that is the graph G_n such that, for every prime p that not divide n, G_n contains one cycles of length p.
3. Let T_g the h-inductive theory of groups in the usual language L_g of groups. T_g is complete and the trivial group is the unique pc model of T_g.
4. Let $L^* = L_g \cup \{R\}$ where L_g is the language of groups and R a symbol of binary relation interpreted by $R(a, b) \leftrightarrow a \neq b$. Let T^*_g the usual theory of groups over the language L_1. Since the L^*-homomorphism are the L_g-embeddings then the pc models of T^*_g are the existentially closed groups in the context of logic with negation, so T^*_g is positively complete.
5. Let $L^+ = L_g \cup \{a\}$ where a is a symbol of constant and let $T^+_g = T_g \cup \{a \neq e\}$. Let p and q two prime numbers. Since the groups $\mathbb{Z}_p$ and $\mathbb{Z}_q$ (where the constant a is interpreted by $\bar{1}$) cannot be L^+-continued in a L^+-group, then the theory T^+_g is not positively complete.
   Let G^+ be a pc group of the theory T^+_g. We claim that G^+ is either simple or the intersection of all nontrivial normal subgroups of G^+ is nonempty. Indeed, suppose that G^+ is not simple and let N be a normal subgroups of G^+. Given that the natural L_g-homomorphism $\pi : G^+ \to G^+/N$ is not an immersion then π is not an L^+-homomorphism, which implies that $\pi(a) = \bar{e}$, so $a \in N$.

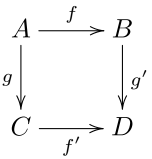
Thereby a belongs to the intersection of all normal subgroups of G^+. Note that if G^+ is simple then G^+ is an existentially closed groups and the constant $a \in L^+$ can be interpreted by any element of $G^+ - \{e\}$. In the case where G^+ is not simple then the constant a can be interpreted by any element of $N - \{e\}$ where N in the intersection of the nontrivial normal subgroups of G^+, and we have $N =< a >^{G^+}$ the normal subgroup generated by a.

2. General forms of positive amalgamation

In this section we will use the letters h, e, i, s to abbreviate the terms respectively of homomorphisms, embeddings, immersions and strong immersions.

DEFINITION 2.1. Let Γ be a class of L-structures and A a member of Γ. We say that A is an:

- $[h, e, i, s]$-amalgamation basis of Γ; if for every B, C in Γ, f an homomorphism from A into B and g an embedding from A into C, there exist $D \in \Gamma$, f' an immersion from B into D and g' a strong immersion from C into D such that the following diagram commutes:



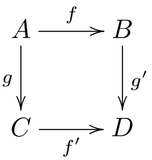
  We say that Γ has the $[h, e, i, s]$-amalgamation property if every element of Γ is an $[h, e, i, s]$-amalgamation basis of Γ.
  By the same way we define all the other possible forms of amalgamation properties.

- $[h, e]$-asymmetric amalgamation basis of Γ, if A is $[h, e, h, e]$-amalgamation basis of Γ.
  By the same way we define all forms of asymmetric amalgamation properties.

- $[h]$- amalgamation basis of Γ, if A is $[h, h, h, h]$-amalgamation basis of Γ.

By the same way we define all the other possible forms of $[x]$-amalgamation properties.

- $[h, e, i, s]$-strong amalgamation basis of Γ, if for every B, C members of Γ such that A is continued into B by an homomorphisms f and embedded in C by an embedding g, then there exist $D \in \Gamma$, f' an immersion from C into D, and g' a strong immersion from B into D such that the following diagram commutes:



  and $\forall (b, c) \in B \times C$, if $g'(b) = f'(c)$ then there is $a \in A$ such that $b = f(a)$ and $c = g(a)$.
  We say that A is an $[h]$- strong amalgamation basis of Γ, if A is a $[h, h, h, h]$- strong amalgamation basis.
  By the same way we define all other possible forms of strong amalgamation properties.

In the following remark, we observe that the most forms of the amalgamations property defined above can be characterized by the notions of completeness and positive completeness defined in the previous section.

Remark 2.2. Let T be an h-inductive L-theory and A a model of T. We have the following properties:

1. A is an $[h]$-amalgamation basis of T if and only if $T \cup Diag^+(A)$ is positively $L(A)$-complete theory.
2. A is an $[e, e, h, h]$-amalgamation basis of T if and only if $T \cup Diag(A)$ is positively $L(A)$-complete theory.
3. A is an $[i, e, h, h]$-amalgamation basis of T if and only if $T \cup Diag^+(A)$ and $T_u(A)$ is are T-complete in the language $L(A)$.

By the same way we can characterize all other forms of amalgamation except the strong amalgamation forms.

In the following example we will list some facts on amalgamation property with the notations and terms given in the definition 2.1.

Example 2.3.

1. Every L-structure A is an $[i, h, s, h]$-amalgamation basis in the class of L-structures (lemma 4, [1]). Since every strong immersion is an immersion, it follows that every L-structure A is an $[s, h]$-asymmetric amalgamation basis in the class of L-structures.

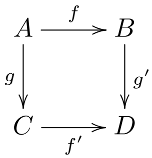
2. Every L-structure A is an $[s, i]$-asymmetric amalgamation basis in the class of L-structures (lemma 5, [1]).

3. Every L-structure A is an $[e, s]$-asymmetric amalgamation basis in the class of L-structures (lemma 4, [2]).

4. Every L-structure A is an $[i, h]$-asymmetric amalgamation basis in the class of L-structures (lemma 8, [5]).

5. Every pc model of an h-inductive theory T is an $[h]$-amalgamation basis in the class of model of T.

LEMMA 2.4. *Let I be a totally ordered set and let $(A_i, f_{i,j})_{i,j\in I}$ be an h-inductive sequence of $[h]$-strong amalgamation basis of an h-inductive theory T. Then the inductive limit of $(A_i, f_{i,j})_{i,j\in I}$ is an h-amalgamation basis of T that satisfies the following property:*
For every models B and C of T, if $f \in Hom(A, B)$ and $g \in Hom(A, C)$ then there is D a model of T such that the following diagram commutes:



where f' and g' are homomorphisms, and $\forall(b, c) \in B \times C$, if $g'(b) = f'(c)$ then there exist $a, a' \in A$ such that $b = f(a)$ and $c = g(a')$.

PROOF: Let A be the h-inductive limit of the sequence $(A_i, f_{i,j})_{i,j\in I}$, let B and C two continuation of A in the class of models of T. We claim that the following set in $L(B \cup C)$-consistent,

$$T \cup Diag^+(B) \cup Diag^+(C) \cup \{b \neq c \mid b \in B - A, c \in C - A\},$$

where every elements of A is interpreted by the same symbols of constant in B and C. Indeed, otherwise there exist $\varphi(\bar{a},\bar{b}) \in Diag^+(B)$ and $\psi(\bar{a},\bar{c}) \in Diag^+(B)$ where $\bar{b} \in B - A$ (ie, if $\bar{b} = (b_1,\cdots,b_n)$ then $\forall 1 \leqslant i \leqslant n, b_i \in B - A$) and $\bar{c} \in C - A$ such that

$$T \vdash \forall \bar{x},\bar{y},\bar{z} \ \ ((\varphi(\bar{x},\bar{y}) \wedge \psi(\bar{x},\bar{z})) \rightarrow \bigvee_{i,j} y_i = z_j) \tag{2.1}$$

Now, let $i \in I$ such that $\bar{a} \in A_i$ and let $f_i = f_{|A_i}$ and $g_i = f_{|A_i}$. Since A_i is a $[h]$-strong amalgamation basis, there are D a model of T, $f' \in Hom(B,D)$ and $g' \in Hom(C,D)$ such that

$$\forall (b,c) \in B \times C,\ f'(b) = g'(c) \rightarrow \exists a \in A\ f_i(a) = b \wedge g_i(a) = c. \tag{2.2}$$

By 2.1 and 2.2, there is $a \in A$ such that $f_i(a) = f(a) = b_i$ and $g_i(a) = g(a) = c_i$, a contradiction. □

Theorem 2.5. *Every L-structure A is a $[s,i,s,i]$-strong amalgamation basis in the class of L-structures.*

Proof: Let A, B and C be three L-structures such that A is immersed in B and strongly immersed in C. Suppose that the set

$$T_i(B) \cup T_u(C) \cup Diag^+(B) \cup Diag^+(C) \cup \{b \neq c |\ b \in B - A, c \in C - A\}$$

is $L(B \cup C)$-inconsistent. Then there are $\neg\psi(\bar{a},\bar{c}) \in T_u(C)$, $\varphi_1(\bar{a},\bar{b}) \in Diag^+(B)$ and $\varphi_2(\bar{a},\bar{c}) \in Diag^+(C)$ where ψ is a positive formula, and φ_1,φ_2 quantifier-free positive formulas, such that:

$$T_i(B) \cup \{\neg\psi(\bar{a},\bar{c}), \varphi_1(\bar{a},\bar{b}), \varphi_2(\bar{a},\bar{c}), \bigwedge_{i,j} b_i \neq c_j\}$$

is $L(B \cup C)$-inconsistent, thereby

$$T_i(B) \vdash \forall \bar{y}((\varphi_1(\bar{a},\bar{b}) \wedge \varphi_2(\bar{a},\bar{y})) \rightarrow (\psi(\bar{a},\bar{y}) \vee \bigvee_{i,j} b_i = y_j)). \tag{2.3}$$

Now, since $C \nvDash \psi(\bar{a},\bar{c})$ and $C \vDash \varphi_2(\bar{a},\bar{c})$, then there is $\bar{a}' \in A$ such that $A \nvDash \psi(\bar{a},\bar{a}')$ and $A \vDash \varphi_2(\bar{a},\bar{a}')$, because otherwise we obtain

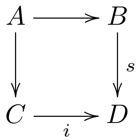
$$A \vdash \forall \bar{x}(\varphi_2(\bar{a},\bar{x}) \rightarrow \psi(\bar{a},\bar{x})).$$

and given that $C \vdash T_i(A)$, we get a contradiction.

So, we obtain $B \nvDash \psi(\bar{a}, \bar{a}')$ and $B \vDash \varphi_2(\bar{a}, \bar{a}')$. From (2.3) we obtain $B \vDash \bigvee_{i,j} b_i = a'_j$, a contradiction. Then

$$T' = T_i(B) \cup T_u(C) \cup Diag^+(B) \cup Diag^+(C) \cup \{b \neq c |\ b \in B - A, c \in C - A\}$$

is $L(B \cup C)$-consistent. Let D a model of T', then the following diagram commutes



where s in a strong immersion and i an immersion. Let $b \in B$ and $c \in C$ such that $s(b) = i(c)$, then there exists $a, a' \in A$ such that $a = b$ and $a' = c$. Thus

$$i(a) = s(a) = s(b) = i(c) = i(a'),$$

So $a = a'$ and $b = c = a$. □

THEOREM 2.6. *Let T be an h-inductive theory. Every model A of T is a $[i, i, h, h]$-strong amalgamation basis of T.*

PROOF: Let A, B and C be models of T. Let f and g two immersions from A to B and C respectively. We claim that the set

$$T_u(A) \cup Diag^+(B) \cup Diag^+(C) \cup \{b \neq c |\ b \in B - A, c \in C - A\}$$

is $L(B \cup C)$-consistent. Indeed, otherwise, there are $\bar{a} \in A, \bar{b} \in B - A, \bar{c} \in C - A$, $\varphi(\bar{a}, \bar{b}) \in Diag^+(B)$, and $\psi(\bar{a}, \bar{c}) \in Diag^+(C)$ such that

$$T_u(A) \cup \{\varphi(\bar{a}, \bar{b}), \psi(\bar{a}, \bar{c}), \bigwedge_{i,j} b_i \neq c_j\}$$

is $L(B \cup C)$-inconsistent, which implies that;

$$T_u(A) \vdash \forall \bar{y}, \bar{z} \ \ ((\varphi(\bar{a}, \bar{y}) \wedge \psi(\bar{a}, \bar{z})) \rightarrow \bigvee_{i,j} y_i = z_j). \tag{2.4}$$

Now, since $C \models \psi(\bar{a}, \bar{c})$ and A is immersed in C, then there is $\bar{a}' \in A$ such that $A \models \psi(\bar{a}, \bar{a}')$, so $B \models \psi(\bar{a}, \bar{a}') \wedge \varphi(\bar{a}, \bar{b})$. On the other hand, given that $\bar{b} \in B - A$ and B is a model of $T_u(A)$ then $B \models \bigvee_{i,j} b_i = a'_j$, a

contradiction.
Let D be a model of $T_u(A) \cup Diag^+(B) \cup Diag^+(C) \cup \{b \neq c |\ b \in B - A, c \in C - A\}$, then the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & B \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[f']{} & D \end{array}$$

where i, i' are immersions and f, f' are homomorphisms. Considering that D is a model of $T_u(A)$ then $f \circ i$ and $f' \circ i'$ are immersions.
Now, let $b \in B$ and $c \in C$ such that $f(b) = f'(c)$, then there exist $a, a' \in A$ such that $i(a) = b$ and $i'(a') = c$. So,

$$f \circ i(a') = f' \circ i'(a') = f(b) = f \circ i(a)$$

then $a = a'$. Thereby A is a $[i, i, h, h]$-strong amalgamation basis of T. $\square$

Corollary 2.7. Every pc model of an h-inductive theory T is an $[h]$-strong amalgamation basis of T.

Lemma 2.8. *Every model of an h-inductive theory T is a $[i, h, s, h]$-strong amalgamation basis of T.*

Proof: Let A, B and C three models of T such that A is immersed in B and continued in C by a homomorphism f. The proof consists in showing the $L(B \cup C)$-consistency of the set

$$T' = T_i(C) \cup Diag^+(B) \cup Diag^+(C) \cup \{b \neq c |\ b \in B - A, c \in C - f(A)\}.$$

Suppose that is not the case, then there are $\varphi(\bar{a}, \bar{c}) \in Diag^+(C)$ and $\psi(\bar{a}, \bar{b}) \in Diag^+(B)$ where $\bar{b} \in B - A$ and $\bar{c} \in C - A$, such that;

$$T_i(C) \vdash \forall \bar{y} \ \ ((\varphi(\bar{a}, \bar{c}) \wedge \psi(\bar{a}, \bar{y})) \rightarrow \bigvee_{i,j} y_i = c_j).$$

Given that $B \models \psi(\bar{a}, \bar{b})$ and A is immersed B, there is $\bar{a}' \in A$ such that $A \models \psi(\bar{a}, \bar{a}')$. Which implies $C \models \varphi(\bar{a}, \bar{c}) \wedge \psi(\bar{a}, \bar{f}(\bar{a}'))$, thereby $C \models \bigvee_{i,j} f(\bar{a}')_i = c_j$, a contradiction.
Let D be a model of T', let f' be the natural homomorphism defined from B into D and i' the natural strong immersion defined from C into D. Let

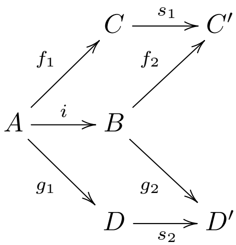
$b \in B$ and $c \in C$ such that $f'(b) = i'(c)$, so there are $a, a' \in A$ such that $i(a) = b$ and $f(a') = c$. Then

$$i' \circ f(a) = f' \circ i(a) = i' \circ f(a'),$$

thus $f(a) = c$ and $i(a) = b$. □

LEMMA 2.9. *Let B be a $[h]$-strong amalgamation basis of T and A a model of T that is immersed in B, then A is a $[h]$-strong amalgamation basis of T.*

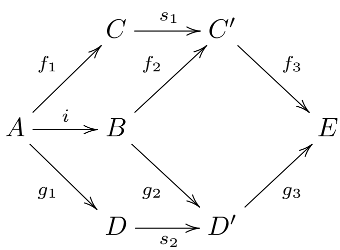
PROOF: Let A, C and D be models of T. Let $f_1 \in Hom(A, B)$ and $f_2 \in Hom(A, D)$. Given that A is immersed in B and every L-structure is an $[i, h, s, h]$-strong amalgamation basis in the class of models of T (Lemma 2.8), we obtain the following commutative diagram:



where i_1 and i_2 are immersions, f_2 and g_2 homomorphisms and C', D' two models of T that satisfy:

$$\begin{cases} \forall (b, c) \in B \times C,\ f_2(b) = s_1(c) \to \exists a \in A,\ b = i(a) \wedge c = f_1(a) \\ \forall (b, d) \in B \times D,\ g_2(b) = s_2(d) \to \exists a \in A,\ b = i(a) \wedge d = g_1(a). \end{cases} \tag{2.5}$$

Now, since B is a $[h]$-strong amalgamation basis of T, we complete the previous diagram and we get the following commutative diagram:



where E is a model of T, f_3 and g_3 two homomorphisms such that:
$\forall c' \in C', \forall d' \in D'$, if $f_3(c') = g_3(d')$ then there is $b \in B$ such that $f_2(b) = c'$ and $g_2(b) = d'$.
Let $c \in C$ and $d \in D$ such that $f_3 \circ s_1(c) = g_3 \circ s_2(d)$, then there is $b \in B$ such that $f_2(b) = s_1(c)$ and $g_2(b) = s_2(d)$. So there are $a, a' \in A$ such that:

$$\begin{cases} f_1(a) = c & i(a) = b \\ g_1(a') = d & i(a') = b, \end{cases}$$

and given that i is an immersion we have $f_1(a) = c$ and $g_1(a) = d$. So, A is a $[h]-$strong amalgamation basis. □

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Positive Complete Theories and Positive Strong Amalgamation Property

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