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THE CARDINAL SQUARING PRINCIPLE AND AN ALTERNATIVE AXIOMATIZATION OF NFU

Abstract

In this paper, we rigorously prove the existence of type-level ordered pairs in Quine's New Foundations with atoms, augmented by the axiom of infinity and the axiom of choice ($\mathsf{NFU} + \mathsf{Inf} + \mathsf{AC}$). The proof uses the cardinal squaring principle; more precisely, its instance for the (infinite) universe (VCSP), which is a theorem of $\mathsf{NFU} + \mathsf{Inf} + \mathsf{AC}$. Therefore, we have a justification for proposing a new axiomatic extension of NFU, in order to obtain type-level ordered pairs almost from the beginning. This axiomatic extension is $\mathsf{NFU} + \mathsf{Inf} + \mathsf{AC} + \mathsf{VCSP}$, which is equivalent to $\mathsf{NFU} + \mathsf{Inf} + \mathsf{AC}$, but easier to reason about.

Keywords: Quine's New Foundations, cardinal multiplication, axiomatization.

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Introduction

Quine's New Foundations (NF) [12] can be viewed as an improved and simplified version of *Principia mathematica*. However, its (relative) consistency has not been proved for thirty years, until Jensen proved that a slight modification of NF admits a consistency proof [11]. Jensen weakened the axiom of extensionality and allowed the atoms to exist in the theory,

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named *New Foundations with atoms* or urelements (NFU). By Jensen's results, NFU is relatively consistent with the axiom of infinity (Inf) and the axiom of choice (AC). So we can work in NFU + Inf + AC, which seems to us a good alternative theory to ZF for the foundation of mathematics. As far as NF (without atoms) is concerned, the search for its consistency proof is still in progress, most prominently by Holmes [8] and Gabbay [4].

Since NFU embodies a kind of type theory, it is important to keep track of (relative) types during syntactic manipulations, the procedure which can sometimes be arduous. The notion that makes this particularly cumbersome is that of ordered pairs. It is impossible to define type-level ordered pairs in NFU alone since their existence implies the axiom of infinity [7], which is independent of NFU [11]. "Type-level" means that an ordered pair has the same type as its components. There are essentially three approaches to deal with type-level ordered pairs "problem" in NFU.

First one can be called *the way of resignation*. In this approach, one simply rejects the necessity of type-level ordered pairs and works with Kuratowski's ones. Although that is a valid approach, Kuratowski's ordered pairs are difficult to work with because the type of an ordered pair is two higher than its components, so "type explosion" happens very soon, and that can be a liability in theory development. For instance, cardinal arithmetic is defined in an unnatural way, and the proofs about it are very cumbersome (see section 4) if one is using Kuratowski's ordered pairs. For that reason, we would prefer to avoid the resignation way.

The second approach is due to Holmes [7] and its main feature is the introduction of a new *axiom of ordered pairs* (which we denote OP) to NFU. This axiom introduces ordered pairs as a primitive notion and in that way enables the existence of type-level ordered pairs of any two entities (sets or atoms). This axiom solves most of our problems and is a good option for theory development. It does have a justification, for it can be proved that inside a model of NFU + Inf one can construct a model of NFU + OP [5]. Nonetheless, its motivation is entirely pragmatical, since it is solely envisioned to solve one technical problem, and introducing ordered pairs as primitives requires extending our language by some (at least one, usually three) function symbols, without intrinsic rules for type assignment. Such an extension changes the notion of atomic formulas, complicating the notion of stratification. Moreover, even though OP is solving a purely technical problem, its ontological commitment is enormous, for it implies the existence of infinitely many arbitrary objects [7] in a non-obvious way.

The third approach is the new one we are proposing: an alternative axiomatic extension of NFU. We have already mentioned that in every model of NFU + Inf one can obtain a model of NFU + OP, but that is only possible by interpreting OP in the signature specifically extended for OP to have its intended meaning. On the contrary, NFU + Inf + AC does not need any artificial signature extension since it can prove VCSP from which type-level ordered pairs can be easily obtained. Therefore, we have a justification to introduce a new axiomatization NFU + Inf + AC + VCSP, which is an axiomatic extension of NFU+Inf+AC. The main advantage of this approach is that ordered pairs are available almost from the start, it is well-motivated, the language does not need to be extended (thus the notion of stratification remains the same), and it does not have any ontological commitment since it is a conservative axiomatic extension. However, there is a need for some Kuratowski's ordered pair-dependent theory, in order to be able to state the new axiom, VCSP. Fortunately, the theory needed for that is rather small (which will be seen soon enough). It is important to note that theory NFU + Inf + VCSP does not prove AC (see theorem 5.3), which means that it cannot be used as a satisfactory theory for development. Therefore, we find NFU + Inf + AC + VCSP to be the best approach proposed so far for the development of set theory in Quine's style.

In order to show that the third approach is a viable option, we need to prove in NFU+Inf+AC (with Kuratowski's ordered pairs) that the cardinal squaring principle holds. From there, we can easily prove the existence of type-level ordered pairs, completing our justification of alternative extension NFU + Inf + AC + VCSP. The cardinal squaring principle has not yet been rigorously proved in our setting, although it seems to be a well-known fact. The main motivation for this paper is a remark about the cardinal squaring principle in [10]. The same remark is also stated in [6] and [5].

Our proof is based on the one in [2]—but it is not the same since we need to take the peculiarities of NFU into consideration. Moreover, our proof that every infinite set has a countable subset, using Kuratowski's ordered pairs, is correct, unlike the ones in contemporary literature. The proof in [7] is using Zorn's lemma on a set of functions from arbitrary subsets of natural numbers to an infinite set, which does not work, for example, for a set of all even natural numbers as a starting infinite set. Moreover, the proof in [9] is using Zorn's lemma on an empty set, if we use any uncountable set as a starting infinite set.

Overview

In the first section, we introduce the necessary syntax and notation. This is done in more detail in [1].

In the second section, we introduce the axioms of NFU, along with some well-known facts needed for the proof of the cardinal squaring principle. Axiom of choice is also introduced in this section, as well as Kuratowski's ordered pairs.

In the third section, we introduce natural and cardinal numbers and state the axiom of infinity. The main result of this section is the proof that every infinite set has a countable subset.

In the fourth section, we introduce the sum and product of cardinal numbers using Kuratowski's ordered pairs. After a few theorems of preparation, we finally prove that the cardinal squaring principle holds in our setting, and then we show how the existence of type-level ordered pairs can be proved.

In the fifth section, we present the resulting axiomatic extension of NFU, as well as some results regarding the mutual provability of various claims we have introduced.

1. Syntax

In this section, we introduce the syntax of NFU as well as some other necessary notions. Most results are stated without proof; the proofs can be found in [1].

An **alphabet** is a collection of:

- (individual) variables $v_0, v_1, \ldots$
- logical symbols (connectives and quantifiers) $\neg$, $\rightarrow$, $\exists$
- non-logical (relation) symbols $\in$, $=$, *set*
- auxiliary symbols (brackets) (,)

Relation symbols $\in$ and $=$ have the usual interpretation, and *set* is a unary relation symbol expressing that an entity is a set. All the other usual logical symbols ($\vee$, $\wedge$, $\leftrightarrow$, $\forall$, $\exists!$) can be defined in terms of the existing ones in the standard way. Formulas are defined in the following way:

$$\varphi ::= x \in y \mid x = y \mid set(x) \mid (\varphi_1 \to \varphi_2) \mid \neg\varphi_1 \mid \exists x\varphi,$$

where x and y denote variables, while φ, φ_1 and φ_2 denote formulas. We will denote with $\varphi(x_1, \dots, x_n)$ that $x_1, \dots, x_n$ are all free variables occuring in φ. We will usually write $(\exists x \in y)\varphi$, $(\forall x \in y)\varphi$, instead of $\exists x(x \in y \wedge \varphi)$ and $\forall x(x \in y \to \varphi)$ respectively. We write $(\forall x, y \in t)\varphi$ as a shorthand for $(\forall x \in t)(\forall y \in t)\varphi$.

A formula φ is **stratified** if there exists a mapping $type_\varphi$ from the variables of φ to the positive natural numbers such that: for every subformula of φ of the form $x = y$, we have $type_\varphi(x) = type_\varphi(y)$, and for every subformula of φ of the form $x \in y$, we have $type_\varphi(y) = type_\varphi(x) + 1$. The number $type_\varphi(x)$ is called the **type** of the variable x in the formula φ. Conditions imposed on the mapping $type_\varphi$ are called **stratification conditions**. We will call mappings satisfying stratification conditions, **type mappings**. Types of variables will be written in superscript.

Definition 1.1. Let $\varphi(x, x_1, \dots, x_n)$ be a stratified formula. An expression of the form $\{z \mid \varphi(z, x_1, \dots, x_n)\}$ is called an **abstraction term**. We extend the notion of (atomic) formulas by allowing them to contain abstraction terms in addition to variables. Formulas containing abstraction terms we call **formulas of the extended language**. Abstraction terms that appear in atomic formulas are eliminated in the following way:

1. $x \in \{z \mid \varphi(z, x_1, \dots, x_n)\} :\Leftrightarrow \varphi(x, x_1, \dots, x_n)$
2. $x = \{z \mid \varphi(z, x_1, \dots, x_n)\} :\Leftrightarrow$
   $set(x) \wedge \forall y\big(y \in x \leftrightarrow y \in \{z \mid \varphi(z, x_1, \dots, x_n)\}\big)$
3. $\{z \mid \varphi(z, x_1, \dots, x_n)\} \in x \;:\Leftrightarrow (\exists y \in x)\big(y = \{z \mid \varphi(z, x_1, \dots, x_n)\}\big)$
4. $set(\{z \mid \varphi(z, x_1, \dots, x_n)\}) :\Leftrightarrow \exists y\big(y = \{z \mid \varphi(z, x_1, \dots, x_n)\}\big)$

Definition 1.2. Let $\varphi(x_1, \dots, x_n, y_1, \dots, y_m)$ and $\psi(w, x_1, \dots, x_n)$ be formulas. Nested abstraction terms are eliminated in the following way:

$$\big\{\{w \mid \psi(w, x_1, \dots, x_n)\} \mid \varphi(x_1, \dots, x_n, y_1, \dots, y_m)\big\} := \big\{z \;\big|\; set(z) \wedge \exists x_1 \cdots \exists x_n\big(\varphi(x_1, \dots, x_n, y_1, \dots, y_m) \wedge z = \{w \mid \psi(w, x_1, \dots, x_n)\}\big)\big\}.$$

It is important to be able to assign types to abstraction terms. If $\varphi(z, z_1, \ldots, z_n)$ is a stratified formula and $\mathsf{t} = \{z \mid \varphi(z, z_1, \ldots, z_n)\}$ is an abstraction term, then the type of t in some stratified formula is determined by the type of a variable t in a formula $z \in t \leftrightarrow \varphi(z, z_1, \ldots, z_n)$. All this has been done more formally and precisely in [1].

When defining sets, we will usually not check whether the defining formulas are stratified. That can be easily done using type assignments in the extended language (again, details about this procedure can be found in [1]). Explicit checking will only be done in some proofs.

2. NFU set theory

Axiom of extensionality:

$$\forall x \forall y \big(set(x) \land set(y) \land \forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y \big).$$

Axiom of sethood:

$$\forall x \, (\forall y \in x) \, set(x).$$

Axiom schema of stratified comprehension: if $\varphi(z, x_1, \ldots, x_n)$ is stratified, then

$$\forall x_1 \cdots \forall x_n \exists y \big(set(y) \land \forall z \big(z \in y \leftrightarrow \varphi(z, x_1, \ldots, x_n) \big) \big).$$

We say that x is a **subset** of y if $set(x) \land set(y) \land (\forall z \in x)(z \in y)$, which is written $x \subseteq y$. We write $x \subset y$ if $x \subseteq y$ and $x \neq y$. Some simple sets and operations in NFU are:

$$\begin{gathered}
\emptyset := \{x \mid x \neq x\}, \qquad V := \{x \mid x = x\}, \qquad SET := \{x \mid set(x)\}, \\
x \cup y := \{z \mid z \in x \lor z \in y\}, \qquad x \cap y := \{z \mid z \in x \land z \in y\}, \\
\bigcup x := \{z \mid (\exists t \in x)(z \in t)\}, \qquad \bigcap x := \{z \mid (\forall t \in x)(z \in t)\}, \\
x \setminus y := \{z \mid z \in x \land z \notin y\}, \qquad x^c := \{z \mid z \notin x\}, \\
\{x\} := \{z \mid z = x\}, \qquad \mathscr{P}_1(x) := \big\{\{t\} \bigm| t \in x\big\}
\end{gathered}$$

It is easy to check the following equivalence:

$$\forall x \big(set(x) \leftrightarrow x = \emptyset \lor (\exists y \in x) \big).$$

Definition 2.1. For $x, y \in V$ we define their **ordered pair** $(x, y) := \big\{\{x\}, \{x, y\}\big\}$ (where $\{a, b\} := \{a\} \cup \{b\}$).

These ordered pairs are the usual Kuratowski's ordered pairs and they have the unfortunate property of not being type-level. More precisely, if x and y have type n in some stratified formula, then (x, y) has type $n+2$ in that same formula.

Definition 2.2. For sets X and Y we define their **Cartesian product** $X \times Y := \{(x, y) \mid x \in X \wedge y \in Y\}$.

Definition 2.3. Let X and Y be sets. We say that R is a **(binary) relation** between X and Y if $R \subseteq X \times Y$, which we write $\mathit{rel}(R, X, Y)$. We say that R is a **relation** if $\mathit{rel}(R, V, V)$.

Let R be a relation. Instead of $(x, y) \in R$ we will write $x \mathrel{R} y$. In addition, if $(x, y) \notin R$, we will write $x \not\mathrel{R} y$.

Definition 2.4. Let R be a relation. We define its **domain** $\mathit{dom}(R) := \{x \mid \exists y(x \mathrel{R} y)\}$ and **range** $\mathit{rng}(R) := \{y \mid \exists x(x \mathrel{R} y)\}$.

Definition 2.5.

1. We define **identity** on a set X as $\mathit{id}_X := \{(x, x) \bigm| x \in X\}$.
2. For relations R and R' we define their **composition** as $R' \circ R := \{(x, z) \bigm| \exists y(x \mathrel{R} y \mathrel{R'} z)\}$.
3. For a relation R, we define its **inverse** as $R^{-1} := \big\{(y, x) \bigm| x \mathrel{R} y\big\}$.

Definition 2.6. Let X and Y be sets. We say that f is a **function** from X to Y if $\mathit{rel}(f, X, Y) \wedge (\forall x \in X)(\exists! y \in Y)(x \mathrel{f} y)$, which we write $\mathit{func}(f, X, Y)$.

For a function f and $x \in \mathit{dom}(f)$, we introduce the standard notation $f(x)$ for the unique y such that $x \mathrel{f} y$.

Definition 2.7. Let X and Y be sets.

1. f is an **injection** from X to Y if $\mathit{func}(f, X, Y) \wedge (\forall x_1, x_2 \in X)$ $\big(f(x_1) = f(x_2) \to x_1 = x_2\big)$, which we write $\mathit{inj}(f, X, Y)$.
2. f is a **bijection** between X and Y if $\mathit{inj}(f, X, Y) \wedge \mathit{rng}(f) = Y$, which we write $\mathit{bij}(f, X, Y)$.

It is easy to see that every relation R is a relation on $dom(R) \cup rng(R)$. We say that R is a **reflexive** relation if $x \mathrel{R} x$ for every $x \in dom(R)$.

DEFINITION 2.8. We say that a relation $\preceq$ is a **partial order** if it is reflexive, antisymmetric and transitive; symbolically, if

$$rel(\preceq, V, V) \wedge \big(\forall x \in dom(\preceq)\big)(x \preceq x) \wedge \\ \wedge \forall x \forall y (x \preceq y \preceq x \rightarrow x = y) \wedge \forall x \forall y \forall z (x \preceq y \preceq z \rightarrow x \preceq z),$$

which is written $Po(\preceq)$. We will say that $\preceq$ is a partial order on a set X if $Po(\preceq) \wedge dom(\preceq) = X$, which is written $Po(\preceq, X)$.

DEFINITION 2.9. We say that a relation R on X is an **equivalence relation** if it is reflexive, symmetric, and transitive. For every $x \in X$ we define its **equivalence class** $[x]_R := \{y \mid y \mathrel{R} x\}$.

DEFINITION 2.10. For a set X we define its **quotient set** by an equivalence relation R on X as $X/R := \{[x]_R \mid x \in X\}$.

DEFINITION 2.11. Let $\preceq$ be a partial order, $Y \subseteq dom(\preceq)$ and $y_0 \in dom(\preceq)$. We say that y_0 is

1. a **maximal** element of Y if $y_0 \in Y \wedge (\forall y \in Y)(y_0 \preceq y \rightarrow y_0 = y)$;
2. an **upper bound** of Y if $(\forall y \in Y)(y \preceq y_0)$;
3. the **greatest** element of Y if $y_0 \in Y$ and y_0 is an upper bound of Y;
4. the **least** element of Y if y_0 is the greatest element of Y with respect to the relation $\succeq := \preceq^{-1}$.

DEFINITION 2.12. Let $\preceq$ be a partial order.
We say that $L \subseteq dom(\preceq)$ is a **chain** in $\preceq$ if $(\forall x, y \in L)(x \preceq y \vee y \preceq x)$.

LEMMA 2.13. *Let X be a set of functions ordered by inclusion, and let $C \subseteq X$ be a chain. Then*

1. $\big(\bigcup C\big)^{-1} = \bigcup\{f^{-1} \mid f \in C\}$.
2. $\bigcup C$ *is a function.*
3. $dom\big(\bigcup C\big) = \bigcup\big\{dom(f) \mid f \in C\big\}$
4. $rng\big(\bigcup C\big) = \bigcup\big\{rng(f) \mid f \in C\big\}$
5. *If every function $f \in C$ is an injection, then $\bigcup C$ is an injection.*

PROOF:

1. Let $z \in \left(\bigcup C\right)^{-1}$, then $z = (x, y)$ for some x, y such that $(y, x) \in \bigcup C$. Then there exists $f \in C$ such that $(y, x) \in f$, which implies $(x, y) \in f^{-1} \subseteq \bigcup\{f^{-1} \mid f \in C\}$. The other direction is analogous.

2. Let $x, y \in dom\big(\bigcup C\big)$. Then there exist $a, b \in rng\big(\bigcup C\big)$ such that $(x, a), (y, b) \in \bigcup C$. It means there exist functions $f_1, f_2 \in C$ such that $(x, a) \in f_1$ and $(y, b) \in f_2$. Because C is a chain, without the loss of generality, we can assume $f_1 \subseteq f_2$. From that we get $(x, a), (y, b) \in f_2 \subseteq \bigcup C$. If $x = y$, since f_2 is a function, we get $a = b$. Therefore, $\bigcup C$ is a function.

3. Let $z \in dom\big(\bigcup C\big)$. Then there exists $y \in rng\big(\bigcup C\big)$ such that $(z, y) \in \bigcup C$. Then there exists a function $f \in C$ such that $(z, y) \in f$, that is, $z \in dom(f)$. From that, we get $z \in \bigcup\big\{dom(f) \mid f \in C\big\}$. If $z \in \bigcup\big\{dom(f) \mid f \in C\big\}$, then there exists $f \in C$ such that $z \in dom(f)$. That means there exists $y \in rng(f)$ such that $(z, y) \in f \subseteq \bigcup C$, which implies $z \in dom\big(\bigcup C\big)$.

4. Follows from (1) and (3).

5. Follows from (1) and (2). □

We say that a relation on X is a **well-order** if it is a partial order and every nonempty subset of X has the least element in that order.

AXIOM OF CHOICE: $\forall x\big(set(x) \land \emptyset \notin x \land (\forall y, z \in x)(y \neq z \rightarrow y \cap z = \emptyset) \rightarrow \exists u(\forall w \in x)\exists! v(v \in w \cap u)\big)$.

***Zorn's lemma*:** Let $\preceq$ be a partial order. If every chain C in $\preceq$ has an upper bound, then $dom(\preceq)$ has a maximal element.

***Zermelo's theorem*:** Every set can be well-ordered.

THEOREM 2.14. *Axiom of choice $\Leftrightarrow$ Zorn's lemma $\Leftrightarrow$ Zermelo's theorem.*

The equivalence proof resembles the usual one (from ZF) and can be found in [7]. It is worth noting that in Zorn's lemma, we can assume that the chain is nonempty provided we prove first that $\preceq$ is nonempty (we can always use any element of $dom(\preceq)$ as an upper bound for the empty chain).

3. Cardinal numbers

Definition 3.1.

1. The set $0 := \{\emptyset\}$ is **zero**.
2. For a set x we define its **successor** $\mathit{succ}(x) := \big\{y \bigm| (\exists z \in y)(y \setminus \{z\} \in x)\big\}$.
3. The set $\mathbb{N} := \bigcap\big\{x \bigm| 0 \in x \wedge (\forall y \in x)\big(\mathit{succ}(y) \in x\big)\big\}$ is called the set of **natural numbers**.
4. The set $\mathit{FIN} := \bigcup \mathbb{N}$ is the set of **finite sets**.

It can be proved that $\mathit{succ}(x) = \big\{y \cup \{z\} \bigm| y \in x \wedge z \notin y\big\}$.
We define $1 := \mathit{succ}(0) = \mathscr{P}_1(V)$ and $2 := \mathit{succ}(1)$.
We say that a set x is **finite** if $x \in \mathit{FIN}$, otherwise it is **infinite**.

Axiom of infinity: $V \notin \mathit{FIN}$.

Peano's axioms are the following:

1. $0 \in \mathbb{N}$.
2. $(\forall n \in \mathbb{N})\big(\mathit{succ}(n) \in \mathbb{N}\big)$.
3. $(\forall n \in \mathbb{N})\big(0 \neq \mathit{succ}(n)\big)$.
4. $(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})\big(\mathit{succ}(n) = \mathit{succ}(m) \rightarrow n = m\big)$.
5. If $\varphi(x)$ is a stratified formula, then

$$\varphi(0) \wedge (\forall n \in \mathbb{N})\big(\varphi(n) \rightarrow \varphi(\mathit{succ}(n))\big) \rightarrow (\forall n \in \mathbb{N})\varphi(n).$$

The first three Peano axioms can be easily proved in NFU. However, the axiom schema of mathematical induction must be restricted to stratified formulas only. Lastly, the fourth Peano axiom is equivalent to the axiom of infinity. This proof can be found in [14], but we will provide one too.

Lemma 3.2. *Every natural number n is 0 or a successor of some natural number.*

Proof: We need to prove $(\forall n^1 \in \mathbb{N}^2)\big(n^1 = 0^1 \vee (\exists m^1 \in \mathbb{N}^2)(n^1 = \mathit{succ}(m^1)^1)\big)$, which is stratified, so we can prove it by induction on n.

The claim trivially holds for $n = 0$. Assume the claim holds for some $n \in \mathbb{N}$ and prove it for $succ(n)$. However, $succ(n)$ is the successor of n, and n is a natural number. $\square$

Definition 3.3. We define the **equipotence** relation between sets as $(\sim) := \{(x, y) \mid set(x) \wedge set(y) \wedge \exists f\, bij(f, x, y)\}$.

Definition 3.4. We define the set of **cardinal numbers** $Card := SET/(\sim) = \{[x]_\sim \mid set(x)\}$.

We will denote $[x]_\sim$ with $|x|$ and call it the **cardinal number of set** x.

Definition 3.5. We define the relation $\leq$ on *Card* with $\kappa \leq \lambda :\Longleftrightarrow (\forall X \in \kappa)(\forall Y \in \lambda)\exists f\, inj(f, X, Y)$.

It is easy to show that it is enough, in definition 3.5, to require just the existence of X and/or Y (the existence of an injection between two sets is invariant with respect to equipotence).

Theorem 3.6. *The relation $\leq$ is a well-order.*

Proof (Sketch of proof:): Reflexivity and transitivity are easy.

The antisymmetricity is actually Cantor–Bernstein's theorem which can be found in [7].

The fact that every nonempty set of cardinals has the least element is proved using Zermelo's theorem. See [7, p. 123–124]. $\square$

We write $\kappa < \lambda$ for $\kappa \leq \lambda \wedge \kappa \neq \lambda$.

Definition 3.7. We say that a set x is **Dedekind-infinite**, if there exists $y \subset x$ such that $x \sim y$.

For now, we can only prove that Dedekind-infinity implies infinity.

Theorem 3.8. *If a set is Dedekind-infinite, then it is infinite.*

Proof: We will prove the contrapositive of the claim: if a set is finite (that is, it is an element of some natural number), then there is no bijection between it and its any proper subset.

Formula $(\forall n^2 \in \mathbb{N}^3)(\forall x^1 \in n^2)\,\forall y^1(y^1 \subset x^1 \to x^1 \not\sim^4 y^1)$ is stratified, so we can prove it by induction on n.

Let $n = 0$ and $x \in n$ be arbitrary. Then $x = \emptyset$, so the statement holds vacuously since x does not have any nonempty proper subsets. Assume that for a natural number n the statement holds. Let us prove the statement for $succ(n)$. Let $x \in succ(n)$ be arbitrary. Then $x = y \cup \{z\}$, for some $y \in n$ and $z \notin y$. Assume that there exists a $u \subset x$ and a bijection $f\colon x \to u$.

First case is when $z \notin u$. Then we have $u \subseteq y$, and so $u \setminus \{f(z)\} \subset y$, but we have $bij(f \setminus \{(z, f(z))\}, y, u \setminus \{f(z)\})$, which is a contradiction.

Second case is when $z \in u$. Then there exists $w \in x$ such that $f(w) = z$. Define $h := id_{x\setminus\{w,z\}} \cup \{(z, w), (w, z)\}$. It is obvious that $bij(h, x, x)$, so $bij(f \circ h, x, u)$. The same argument holds if $z = w$. Since $(f \circ h)(z) = f(w) = z$, we have $bij(f \setminus \{(z, z)\}, y, u \setminus \{z\})$. However, we have $u \setminus \{z\} \subset y$, which is a contradiction. □

Lemma 3.9. *The following holds:* $(\forall n \in \mathbb{N})(\forall y \in n)\forall z(z \in n \leftrightarrow y \sim z)$.

Proof: Formula $(\forall n^2 \in \mathbb{N}^3)(\forall y^1 \in n^2)\forall z^1(z^1 \in x^2 \leftrightarrow z^1 \sim^4 y^1)$ is stratified, so we can prove it by induction on x.

Let $n = 0$, let $y \in n$ and z be arbitrary. From $y \in n$ we get $y = \emptyset$. If $z \in n$, we get $z = \emptyset = y$. Obviously, $bij(\emptyset, \emptyset, \emptyset)$. If $y \sim z$, then there exists a bijection f between y and z. Because $y = \emptyset$, f is a bijection between $\emptyset$ and z, so we have $z = rng(f) = rng(\emptyset) = \emptyset \in 0 = n$.

Assume that the claim holds for a natural number n. Let us prove it for $succ(n)$. If $succ(n) = \emptyset$, the claim trivially holds. Let $y \in succ(n)$ and z be arbitrary. By the definition of successor we have $y = a \cup \{b\}$ such that $a \in n$ and $b \notin a$. If $z \in succ(n)$, then $z = u \cup \{v\}$, where $u \in n$ and $v \notin u$. By the assumption, we have $u \sim a$, so there exists a bijection $f\colon a \to u$ and the function $g := f \cup \{(b, v)\}$ is obviously a bijection between y and z. If $y \sim z$, then there exists a bijection $f\colon y \to z$. We define $x := \{f(t) \mid t \in a\}$ and $y := f(b)$. Obviously, $y \notin x$ and since $x \sim a \in n$, we have $z = x \cup \{y\} \in succ(n)$. □

Theorem 3.10. $V \notin \mathit{FIN} \iff \emptyset \notin \mathbb{N} \iff \mathbb{N} \subseteq \mathit{Card} \iff$ (P4).

Proof: We will first prove $V \notin \mathit{FIN} \implies \emptyset \notin \mathbb{N}$. Assume $V \notin \mathit{FIN}$. Formula $(\forall n^1 \in \mathbb{N}^2)(n^1 \neq \emptyset^1)$ is stratified, so we can prove it by induction on n. For $n = 0$ we have $0 = \{\emptyset\} \neq \emptyset$. Let us assume that the claim holds for an $n \in \mathbb{N}$ and prove it for $succ(n)$. Assume that $succ(n) = \emptyset$. We know $succ(n) = \big\{y \cup \{z\} \bigm| y \in n \wedge z \notin y\big\} = \big\{t \mid (\exists y \in n)(\exists z \notin y)(t = y \cup \{z\})\big\}$. Now from $succ(n) = \emptyset$ follows $(\forall y \in n)\forall z(z \in y)$, and from the axiom of extensionality and the fact that V is the universal set we get $\forall y\big(y \in n \to$

$V = y\big)$. From that we get $n = \emptyset$ or $n = \{V\}$. It is impossible to have $n = \emptyset$ by the induction hypothesis, and from $n = \{V\}$ we get $V \in FIN$, which contradicts the assumption.

The claim $\emptyset \notin \mathbb{N} \implies \mathbb{N} \subseteq Card$ follows from lemma 3.9. Assume $\emptyset \notin \mathbb{N}$. Formula $(\forall n^1 \in \mathbb{N}^2)(n^1 \in Card^2)$ is stratified, so we can prove it by induction on n. For $n = 0$ we have $0 = |\emptyset| \in Card$. Assume that for an $n \in \mathbb{N}$ there exists x such that $n = |x|$, and prove the claim for $succ(n)$. By the assumption, we have $succ(n) \neq \emptyset$, so there exists a $y \in succ(n)$. By lemma 3.9 we have $z \in succ(n) \leftrightarrow z \in |y|$. Therefore, $succ(n) = |y|$.

Let us prove $\mathbb{N} \subseteq Card \implies$ (P4). Assume $\mathbb{N} \subseteq Card$. Let $n, m \in \mathbb{N} \subseteq Card$ and assume $succ(n) = succ(m) = |z|$ for some z. By definitions of successor we have $z = a \cup \{b\} = c \cup \{d\}$, where $a \in n$, $c \in m$, $b \notin a$ and $d \notin c$. If $b = d$, then we have $a = c$, for if $w \in a \subseteq z = c \cup \{d\}$, then $w \in c \cup \{d\}$, and since $w \neq d$, then $w \in c$. The converse when $w \in c$ is proved analogously. If $b \neq d$, then $d \in a$ and $b \in c$. Then $g := id_{z \setminus \{b,d\}} \cup \{(b, d)\}$ is obviously a bijection between a and c. In both cases, we have $a \sim c$, meaning $n = m$.

It remains to prove (P4) $\implies V \notin FIN$. Assume $V \in FIN$. Then there exists an $n \in \mathbb{N}$ such that $V \in n$. We claim $n = \{V\}$. Assume $x \in n$ such that $x \neq V$. Then we have $x \subset V$ and because $x, V \in n$, by lemma 3.9 we get $x \sim V$. Now from theorem 3.8 we get that V is an infinite set, that is, $V \notin FIN$, which is a contradiction. Therefore, $n = \{V\}$. Now by the definition of successor we have $succ(n) = \{y \cup \{z\} \mid y \in n \wedge z \notin y\} = \{y \cup \{z\} \mid z \notin V\} = \emptyset \in \mathbb{N}$ and $succ(\emptyset) = \emptyset$. Then we have $succ(n) = succ(\emptyset)$, but clearly $n \neq \emptyset$. Therefore, fourth Peano axiom does not hold. □

We will say that a cardinal number is an **infinite cardinal number** if it is not a natural number (that is, if it is $|X|$ for an infinite X).

THEOREM 3.11. *Let X be a set, $x_0 \in X$, and $f\colon X \to X$ be a function. Then there exists a unique function $g\colon \mathbb{N} \to X$ such that $g(0) = x_0$ and $g\big(succ(n)\big) = f\big(g(n)\big)$ for every $n \in \mathbb{N}$.*

PROOF: Let X, x_0 and f be as stated. Formula

$$\varphi(t) := \big((0^1, x_0^1)^3 \in t^4 \wedge (\forall n^1 \in \mathbb{N}^2)(\forall y^1 \in X^2)$$
$$\big((n^1, y^1)^3 \in t^4 \to (succ(n^1)^1, f^4(y^1)^1)^3 \in t^4\big)\big)$$

is stratified, so we can define the set $S := \{t \mid \varphi(t)\}$. Obviously $\mathbb{N} \times X \in S$, so $g := \bigcap S \subseteq \mathbb{N} \times X$. In other words, we have $\mathit{rel}(g, \mathbb{N}, X)$, so we can use the usual infix notation for g.

First, we prove $\varphi(g)$. For every $t \in S$ we have $\varphi(t)$, so $0\ t\ x_0$, therefore also $0\ g\ x_0$. In the same manner, if $n \in \mathbb{N}$, $y \in X$ and $n\ g\ y$, then for every $t \in S$ we have $n\ t\ y$. By $\varphi(t)$, we have $\mathit{succ}(n)\ t\ f(y)$, and therefore also $\mathit{succ}(n)\ g\ f(y)$. So, we can conclude $g \in S$.

Formula $\psi(n) := (\exists!\, y^1 \in X^2)(n^1 g^4 y^1)$ is stratified, so we can prove $(\forall n \in \mathbb{N})\psi(n)$ by induction (fifth Peano axiom). For $n = 0$ we have $0\ g\ x_0$ by $\varphi(g)$, and for any $x' \in X \setminus \{x_0\}$ we can see that $g' := g \setminus \{(0, x')\}$ also satisfies φ. Namely, $(0, x')$ is different from any ordered pair forced into t by φ, since $x' \neq x_0$, and $0 \neq \mathit{succ}(n)$ for any n by third Peano axiom. Therefore we have $g' \in S$, from which $g \subseteq g' = g \setminus \{(0, x')\}$, hence $0 \not g\ x'$.

In much the same manner, suppose that (for a particular $k \in \mathbb{N}$) there is a unique $y \in X$ such that $k\ g\ y$. By $\varphi(g)$ we have $\mathit{succ}(k)\ g\ f(y)$, so existence holds. To prove uniqueness, suppose $\mathit{succ}(k)\ g\ u$ for some $u \in X \setminus \{f(y)\}$. Now we prove that $g'' := g \setminus \{(\mathit{succ}(k), u)\}$ satisfies φ: it obviously satisfies the first conjunct since $0 \neq \mathit{succ}(k)$ by third Peano axiom. For the second, let $m \in \mathbb{N}$ and $z \in X$ be such that $m\ g''\ z$. Then $m\ g\ z$ since $g'' \subseteq g$, and therefore $\mathit{succ}(m)\ g\ f(z)$. But if $\big(\mathit{succ}(m), f(z)\big) = \big(\mathit{succ}(k), u\big)$, then $m = k$ by fourth Peano axiom, and also $u = f(z)$. However, $u \neq f(y)$ means $y \neq z$, and that contradicts uniqueness for k.

Therefore we have $\mathit{func}(g, \mathbb{N}, X)$, so we can use the usual function notation for g. We have already proved $g(0) = x_0$. For any $n \in \mathbb{N}$, we have $n\ g\ g(n)$ (since $\mathit{dom}(g) = \mathbb{N}$), and from $\varphi(g)$ we also have $\mathit{succ}(n)\ g\ f\big(g(n)\big)$, which in function notation is exactly $g\big(\mathit{succ}(n)\big) = f\big(g(n)\big)$.

It remains to prove that such g is unique. Assume the opposite, that there is $h \colon \mathbb{N} \to X$ such that $h \neq g$, $h(0) = x_0$, and for all $n \in \mathbb{N}$, $h\big(\mathit{succ}(n)\big) = f\big(h(n)\big)$. Formula $(\forall n^1 \in \mathbb{N}^2)\big(h^4(n^1)^1 = g^4(n^1)^1\big)$ is stratified, so we can prove it by induction on n. For $n = 0$ we have $h(0) = x_0 = g(0)$. Assume that the claim holds for some $n \in \mathbb{N}$, and prove it for $\mathit{succ}(n)$. We have $h\big(\mathit{succ}(n)\big) = f\big(h(n)\big) = f\big(g(n)\big) = g\big(\mathit{succ}(n)\big)$. Therefore, $h = g$, which is a contradiction. $\square$

It is easy to see that, in theorem 3.11, if x_0 has type k, X has type $k+1$, and f has type $k+3$, then g has type $k+3$.

THEOREM 3.12. *Let* $\mathfrak{t}$ *be a term with variable* x *free, which is type-level—that is, in every stratified formula in the extended language where* $\mathfrak{t}$ *appears,* $type(\mathfrak{t}) = type(x)$*. Then for every set* A*, there is a function* f *such that*

$$f(x) = \mathfrak{t}, \qquad \text{for every } x \in A. \tag{3.1}$$

Furthermore, f *is unique if we require additionally that* $dom(f) = A$*.*

That is, we can *define* a function by expression (3.1). We write shorthand "$f(x) := \mathfrak{t},\ x \in A$" and call it the *definition of a function from a domain and a type-level term.* We treat A as a constant—it can also be a variable, but it must not appear in $\mathfrak{t}$ then.

PROOF: We define $f := \{(x, \mathfrak{t}) \mid x \in A\} = \{p \mid (\exists x^1 \in A^2)(p^3 = (x^1, \mathfrak{t}^1)^3)\}$. Since $\mathfrak{t}$ is type-level, this is well-defined, and the existence (and sethood) of f follows from the axiom of stratified comprehension. The uniqueness follows from extensionality. □

In a completely analogous way, we can prove the following.

COROLLARY 3.13. Let $\mathfrak{t}$ be a term with variables x and y free, such that in every stratified formula in the extended language where $\mathfrak{t}$ appears, $type(\mathfrak{t}) = type((x, y))$. Then for every two sets A and B, there is a function f such that

$$f(x, y) = \mathfrak{t}, \qquad \text{for every } x \in A \text{ and } y \in B. \tag{3.2}$$

Furthermore, f is unique if we require additionally that $dom(f) = A \times B$.

Besides the infinitude of V, stated in the axiom of infinity, we can now prove the infinitude of another set.

THEOREM 3.14. *The set* $\mathbb{N}$ *is Dedekind-infinite.*

PROOF: Since $succ(n)$ is type-level in n, by theorem 3.12 there is a function s such that $s(n) := succ(n),\ n \in \mathbb{N}$. By second Peano axiom s is a function from $\mathbb{N}$ to $\mathbb{N}$, and it is an injection because of fourth Peano axiom. Therefore, $bij\big(s, \mathbb{N}, rng(s)\big)$, and from third and first Peano axiom, $rng(s) \subseteq \mathbb{N} \setminus \{0\} \subset \mathbb{N}$. □

The fact that $\mathbb{N}$ is infinite is an easy consequence of theorems 3.8 and 3.14. The cardinal number of natural numbers is $\aleph_0 := |\mathbb{N}|$. We say that a set X is **countable** if $|X| = \aleph_0$.

From theorems 3.10 and 3.11 we can define the sum and product of natural numbers. Note that *succ* can be viewed as a function on the set of natural numbers, which we denote by s as in the proof of theorem 3.14. For an arbitrary $m \in \mathbb{N}$ by theorem 3.11 there exists a function s_m such that $s_m(0) = m$ and $s_m\big(succ(n)\big) = s\big(s_m(n)\big)$ for every $n \in \mathbb{N}$. Since s_m has the same type as s, it is easy to see that if m has a type k, then s_m has a type $k+3$, because m plays the role of x_0 from theorem 3.11. We now define the **sum of natural numbers** as a function term $add(m, n) := s_m(n)$. Obviously, if m and n have type k, then $s_m(n)$ has type k. We use the usual notation for summation $m + n := add(m, n)$.

The product of natural numbers is defined in the same way. For arbitrary $m \in \mathbb{N}$ by theorem 3.11 there exists a function p_m such that $p_m(0) = 0$ and $p_m\big(succ(n)\big) = s_m\big(p(n)\big)$. We now define the **product of natural numbers** $n, m \in \mathbb{N}$ as a function term $mul(m, n) := p_m(n)$. Similarly (since s_m plays the role of f from theorem 3.11), if m and n have type k, then $mul(m, n)$ has a type k. We use the usual notation for multiplication $m \cdot n := p_m(n)$. It is easy to prove by induction the usual properties of addition and multiplication.

Lemma 3.15. *If X is a finite set, and $Y \subseteq X$, then Y is finite.*

Proof: Let X be an arbitrary finite set, that is, $X \in FIN$. That means there exists $n \in \mathbb{N}$ such that $X \in n$. It is enough to prove the formula $(\forall n^2 \in \mathbb{N}^3)(\forall X^1 \in n^2)\forall Y^1(Y^1 \subseteq X^1 \to Y^1 \in FIN^2)$, which is stratified, so we can prove it by induction on n.

If $n = 0$, we have $X \in 0$. That implies $X = \emptyset$. Now for $Y \subseteq \emptyset$ we have $Y = \emptyset \in 0 \subseteq FIN$. Let us assume the claim holds for some n and prove it for $succ(n)$.

Let $X \in succ(n)$ and $Y \subseteq X$. That means $X = x \cup \{z\}$, for some $x \in n$ and $z \notin x$. If $z \notin Y$, then $Y \subseteq x \in n$, and from the induction assumption we have $Y \in FIN$. If $z \in Y$, then $Y \setminus \{z\} \subseteq x$, so from the induction assumption we have $Y \setminus \{z\} \in FIN$, that is, there exists some $k \in \mathbb{N}$ such that $Y \setminus \{z\} \in k$. Because $z \notin Y \setminus \{z\}$, we have $Y = Y \setminus \{z\} \cup \{z\} \in succ(k) \subseteq FIN$ by definition of successor. □

Lemma 3.16. *The following statements hold:*

1. *For all $n \in \mathbb{N}$, if $x \in succ(n)$ and $y \in x$, then $x \setminus \{y\} \in n$.*

2. *There is no natural number n such that $n < 0$.*

3. *For all $n \in \mathbb{N}$ we have $n < succ(n)$.*
4. *For every $n, m \in \mathbb{N}$, $m \leq n$ if and only if $m < succ(n)$.*
5. *Every nonempty partial ordered finite set has a maximal element.*
6. *For all $n \in \mathbb{N}$ we have $n < \aleph_0$.*
7. *For $n, m \in \mathbb{N}$, if $n < m$, then $succ(n) < succ(m)$.*
8. *For $n, m \in \mathbb{N}$, if $n < m$, then for every $x \in \mathbb{N}$, $n + x < m + x$.*
9. *For $n, m \in \mathbb{N}$, if $n < m$, then for every $x \in \mathbb{N}$, $n \cdot x \leq m \cdot x$.*

PROOF:

1. Let $n \in \mathbb{N}$ be arbitrary, $x \in succ(n)$ and $y \in x$. Because $x \in succ(n)$, there exists $z \in x$ such that $x \setminus \{z\} \in n$. Then one bijection between $x \setminus \{y\}$ and $x \setminus \{z\}$ is given by $id_{x \setminus \{y\}}$ if $y = z$, and by $id_{x \setminus \{y,z\}} \cup \{(z, y)\}$ otherwise.

2. Assume the contrary, that there exists $n \in \mathbb{N}$ such that $n \leq 0 \wedge n \neq 0$. Let $A \in n$ and $B \in 0$ be arbitrary. By definition of relation $\leq$, there exists an injection from A to B. However, $B \in 0$ means $B = \emptyset$, therefore, that injection must be empty, hence $A = \emptyset$. That is impossible because $n \neq 0$.

3. Let $A \in n \in \mathbb{N}$ be arbitrary. Then $A \neq V$, so there exists $x \in V$ such that $x \notin A$. Now by the characterization of a successor, we have $A \cup \{x\} \in succ(n)$. Obviously $inj(id_A, A, A \cup \{x\})$, so $n \leq succ(n)$. Assume that $n = succ(n)$. Then there exists a bijection $f\colon A \to A \cup \{x\}$. Obviously $A \subset A \cup \{x\}$, and by theorem 3.8 we have that $A \cup \{x\}$ is infinite, therefore $A \cup \{x\} \notin FIN$. But we also have $A \cup \{x\} \in succ(n) \subseteq FIN$, which means that the assumption is wrong, therefore $n < succ(n)$.

4. Let n and m be arbitrary. Assume $m \leq n$. From 3.16(3) we have $n \leq succ(n)$ and $n \neq succ(n)$. From transitivity of relation $\leq$ we get $m \leq succ(n)$. Assume $m = succ(n)$. Then we have $succ(n) \leq n$ and $n \leq succ(n)$, which gives $n = succ(n)$. That is a contradiction with $n \neq succ(n)$, so $m \leq succ(n)$ and $m \neq succ(n)$, that is, $m < succ(n)$.

   Assume $m < succ(n)$. Let $A \in m$ and $B \in succ(n)$ be arbitrary. By the assumption, there exists an injection $f\colon A \to B$, which is not a bijection. That means there exists $b \in B$ such that $b \notin \mathit{rng}(f)$. That implies f is also an injection from A to $B \setminus \{b\}$, and because lemma 3.16(1) implies $B \setminus \{b\} \in n$, we get $m \leq n$.

5. We need to prove

$$(\forall X \in \mathit{FIN} \setminus \{\emptyset\})\forall R\big(Po(R, X) \rightarrow \\ \rightarrow (\exists x_0 \in X)(\forall y \in X \setminus \{x_0\})(x_0 \not R\, y)\big),$$

which follows from the stratified formula

$$(\forall n^3 \in \mathbb{N}^4 \setminus \{0^3\}^4)(\forall X^2 \in n^3)\forall R^4\big(Po(R^4, X^2) \rightarrow \\ \rightarrow (\exists x_0^1 \in X^2)(\forall y^1 \in X^2 \setminus \{x_0^1\}^2)(x_0^1 \not R^4 y^1)\big).$$

We will prove it by induction. Let $n = 1$, $X \in 1$ be arbitrary and R be a partial order on X. From $X \in 1 = \mathit{succ}(0)$ we get that there exists z_0 such that $X = \{z_0\}$, and then $R = \{(z_0, z_0)\}$, so z_0 is a maximal element of X under relation R. Assume the claim holds for a natural number $n \geq 1$, and prove it for $\mathit{succ}(n)$.

Let $X \in \mathit{succ}(n)$ be arbitrary, and R be a partial order on X. From characterization of successor, we have $X = x \cup \{y\}$, where $x \in n$ and $y \notin x$. Since R is a partial order, $R' := R \cap (x \times x)$ is partial order on x. From the induction hypothesis, we have that there exists a maximal element z_0 of x under relation R'. If $z_0\; R\; y$, then y must be a maximal element of X under R. For if there existed some $w_0 \neq y$ such that $y\; R\; w_0$, then from $z_0\; R\; y$ and transitivity of R, we would have $z_0\; R\; w_0$. Since $w_0 \neq y$, we have $w_0 \in x$, which is a contradiction with maximality of z_0 in x. If $y\; R\; z_0$, or z_0 and y are not comparable, then z_0 is a maximal element of X under R.

6. Let $n \in \mathbb{N}$ be arbitrary. Assume $n \geq \aleph_0$. By definition of $\leq$, there exists and injection $f \colon \mathbb{N} \rightarrow A$, where $A \in n$. That means $\mathbb{N} \sim \mathit{rng}(f)$, and also $\mathit{rng}(f) \subseteq A$. Because A is finite, by theorem 3.15 we have $\mathit{rng}(f)$ finite, which implies that $\mathbb{N}$ is finite, which is a contradiction. Therefore, $n < \aleph_0$.

7. Take $n, m \in \mathbb{N}$ such that $n < m$. Assume $\mathit{succ}(m) \leq \mathit{succ}(n)$. By (3) we have $m < \mathit{succ}(m) \leq \mathit{succ}(n)$, and from (4) we get $m \leq n$, which is a contradiction.

8. Take arbitrary $n, m \in \mathbb{N}$. Formula $(\forall x^1 \in \mathbb{N}^2)(n^1 <^4 m^1 \to (m^1 + x^1)^1 <^4 (n^1 + x^1)^1)$ is stratified, so we can prove it by induction on x. If $x = 0$, then from $n < m$ we get $n + x = n + 0 = n < m = m + x$. Assume that the claim holds for some $x \in \mathbb{N}$, and let us prove it for $succ(x)$. If $n < m$, then from the associativity we get $n + succ(x) = succ(n + x)$, and by the induction assumption, we have $n + x < m + x$. Now from (7) we get $succ(n + x) < succ(m + x)$. Therefore, $n + succ(x) \leq succ(m + x) = m + succ(x)$.

9. Take arbitrary $n, m \in \mathbb{N}$. Formula $(\forall x^1 \in \mathbb{N}^2)\big(n^1 <^4 m^1 \to (n^1 \cdot x^1)^1 \leq^4 (m^1 \cdot x^1)^1\big)$ is stratified, so we can prove it by induction on x. If $x = 0$, then the claim trivially holds. Let us assume the claim for some $x \in \mathbb{N}$, and prove it for $succ(x)$. If $n < m$, then from the induction assumption, (8) and commutativity of addition, we get $n \cdot succ(x) = n \cdot x + n \leq m \cdot x + m = m \cdot succ(x)$. □

Remark 3.17. It is useful to note that if the partial order in lemma 3.16(5) is a well-order, then a maximal element is also the greatest element.

Definition 3.18. For every $n \in \mathbb{N}$ we define its initial segment as

$$A_n := \{m \in \mathbb{N} \mid m < n\}.$$

Note that if n has type s, then A_n has type $s + 1$.

Lemma 3.19.

1. *For every $n \in \mathbb{N}$, the set A_n is finite.*

2. *If X is a set of initial segments of natural numbers and $\bigcup X \subset \mathbb{N}$, then $\bigcup X$ is an initial segment of natural numbers.*

Proof:

1. Formula $(\forall n^1 \in \mathbb{N}^2)(A^2_{n^1} \in FIN^3)$ is stratified, so we can perform mathematical induction on n. For $n = 0$ we have $A_0 = \{m \in \mathbb{N} \mid m < 0\} = \emptyset \in 0 \subseteq FIN$. Assume that for some $n \in \mathbb{N}$, set A_n is finite, and let us prove the statement for $A_{succ(n)}$. Because $A_n \in FIN$, there exists $k \in \mathbb{N}$ such that $A_n \in k$ and there exists $x \notin A_n$ such that $A_n \cup \{x\} \in succ(k)$. From lemma 3.16(4) we have $A_{succ(n)} = A_n \cup \{n\}$,

therefore $id_{A_n} \cup \{(x, n)\}$ is a bijection between $A_n \cup \{x\}$ and $A_n \cup \{n\}$, which means $A_n \cup \{n\} = A_{n+1} \in succ(k) \subseteq FIN$.

2. Assume that X is a set of initial segments of natural numbers such that $\bigcup X \subset \mathbb{N}$. If $X = \{A_0\}$, then obviously $\bigcup X = A_0$, so assume $X \neq \{A_0\}$. The set $\bigcup X$ is a proper subset of $\mathbb{N}$, so there is an $m \in \mathbb{N}$ such that $m \notin \bigcup X$. Then $\bigcup X \subseteq A_m$, so by (1) and lemma 3.15, $\bigcup X$ is a nonempty finite subset of $\mathbb{N}$. Therefore, by remark 3.17, it has the greatest element r. Then for every $x \in \bigcup X$, $x \leq r$, therefore by lemma 3.16(4) $x < succ(r)$. So, $\bigcup X \subseteq A_{succ(r)}$. For the opposite inclusion, suppose $x \in A_{succ(r)}$. Since $r \in \bigcup X$, there is $A_i \in X$ such that $r \in A_i$. Then $x \leq r < i$ implies $x \in A_i \subseteq \bigcup X$. □

The following theorem is very important for accomplishing our goal.

Theorem 3.20. *Every infinite set has a countable subset.*

Proof: Let X be an infinite set. We will prove that there is an injection from $\mathbb{N}$ to X.

Formula $(\exists n^1 \in \mathbb{N}^2) inj(f^4, A^2_{n^1}, X^2) \vee inj(f^4, \mathbb{N}^2, X^2)$ is stratified, so we can define a set $K := \big\{ f \mid (\exists n \in \mathbb{N}) inj(f, A_n, X) \vee inj(f, \mathbb{N}, X) \big\}$. Set K is nonempty because for $n = 0$ we have $A_0 = \emptyset$ by lemma 3.16(2), which means $inj(\emptyset, A_0, X)$. We order K by inclusion and prove that it satisfies the remaining condition of Zorn's lemma.

Let $C \subseteq K$ be an arbitrary nonempty chain. We need to prove $\bigcup C \in K$. From lemma 2.13 we get that $\bigcup C$ is an injection, $rng\big(\bigcup C\big) \subseteq X$, and $dom\big(\bigcup C\big) \subseteq \mathbb{N}$. If $dom\big(\bigcup C\big) \neq \mathbb{N}$, then since the domain of every element of C is an initial segment of natural numbers, from lemma 2.13 and lemma 3.19(2) there exists some $n_0 \in \mathbb{N}$ such that $dom\big(\bigcup C\big) = A_{n_0}$, which implies $\bigcup C \in K$. If $dom\big(\bigcup C\big) = \mathbb{N}$, then obviously $\bigcup C \in K$. Now from Zorn's lemma, there exists a maximal element of K, which we denote by f_0.

If $dom(f_0) \neq \mathbb{N}$, then there exists $n \in \mathbb{N}$ such that $dom(f_0) = A_n$. If $rng(f_0) = X$, we have $bij(f_0, A_n, X)$, which is a contradiction because A_n is finite by lemma 3.19(1) and X is infinite by assumption. If $rng(f_0) \neq X$, then there exists $x \in X \setminus rng(f_0)$. Define the function $F := f_0 \cup \{(n, x)\}$. Obviously $f_0 \subset F \in K$, which is a contradiction with the maximality of f_0. Therefore, $dom(f_0) = \mathbb{N}$. So, we have $inj(f_0, \mathbb{N}, X)$, and $rng(f) \sim \mathbb{N}$ is a desired countable subset of X. □

THEOREM 3.21. *If $X \subseteq \mathbb{N}$ is an infinite set, then X is countable.*

PROOF: From $X \subseteq \mathbb{N}$ we have $|X| \leq \aleph_0$. On the other hand, X is an infinite set so by theorem 3.20 there exists $X_0 \subseteq X$ such that $|X_0| = \aleph_0$. But now we have $\aleph_0 = |X_0| \leq |X|$, and because $\leq$ is antisymmetric, that means $|X| = \aleph_0$. □

It is now easy to prove that infinity implies Dedekind-infinity, but this result is not needed for our purposes.

4. The cardinal squaring principle

DEFINITION 4.1. Let κ and λ be cardinal numbers. We define their **level sum** as $\kappa +_L \lambda := \{z \mid (\exists x \in \kappa)(\exists y \in \lambda)(x \times \{0\} \cup y \times \{1\} \sim z \times \{2\})\}$ and their **level product** as $\kappa \cdot_L \lambda := \{z \mid (\exists x \in \kappa)(\exists y \in \lambda)(x \times y \sim z \times \{2\})\}$.

These two operations are defined in such a way as to assure that their types are the same as the types of their operands.

Remark 4.2. **However**, they do not necessarily capture what we expect of the sum and product of cardinal numbers. More precisely, their results don't have to be cardinal numbers. In order for $\kappa \cdot_L \lambda$ to be a cardinal number, it must be nonempty, therefore there must exist $x \in \kappa$, $y \in \lambda$ and z such that $x \times y \sim z \times \{2\}$. But if in particular $\kappa := \lambda := |V|$, then we must have

$$V \times V \sim x \times y \sim z \times \{2\} \subseteq V \times \{2\} \subseteq V \times V,$$

and therefore by Cantor–Bernstein's theorem $V \times V \sim V \times \{2\}$, which is equivalent to VCSP. The other direction is even easier: if VCSP holds, then for every two cardinals κ and λ, for every $x \in \kappa$ and $y \in \lambda$, we can restrict the bijection between $V \times V$ and $\mathscr{P}_1^2(V)$ to $x \times y$, and its image obviously must be of the form $\mathscr{P}_1^2(z)$ for some z. The same bijection can also be restricted to $x \times \{0\} \cup y \times \{1\}$, giving us the nonemptiness of $\kappa + \lambda$.

So, definition 4.1 really defines binary operations on *Card* if and only if VCSP holds. While itself a good motivation for the inclusion of VCSP as an axiom, this argument shows that we must define the cardinal sum and product differently in order to be able to prove VCSP. We will define the aforementioned operations in the usual way, but some claims will then be stated with type raising operation T.

Definition 4.3. For every $\kappa = |x| \in \mathit{Card}$, we define $T(\kappa) := |\mathscr{P}_1(x)|$.

It is important to note that $T(\kappa)$ for a cardinal number κ does not depend on the representative $x \in \kappa$. It is immediate from the definition that if κ has type n, then $T(\kappa)$ has type $n+1$. We also define $T^0(\kappa) := \kappa$ and $T^{k+1}(\kappa) = T\big(T^k(\kappa)\big)$. It easily follows from the definition that if $T(\kappa) = T(\lambda)$ for some cardinal numbers κ and λ, then $\kappa = \lambda$.

In addition, we introduce the symbol for singleton ι with $\iota^0(x) := x$ and $\iota^{k+1}(x) := \{\iota^k(x)\}$. Obviously, if x has type n, then $\iota^k(x)$ has type $n+k$.

Definition 4.4. For cardinal numbers κ and λ, we define their **outer sum** and **outer product** as

$$\kappa \oplus \lambda := \big\{z \bigm| (\exists x \in \kappa)(\exists y \in \lambda)(z \sim x \times \{0\} \cup y \times \{1\})\big\},$$
$$\kappa \odot \lambda := \{z \mid (\exists x \in \kappa)(\exists y \in \lambda)(z \sim x \times y)\}.$$

If κ and λ have type n, then $\kappa \oplus \lambda$ and $\kappa \odot \lambda$ have type $n+2$.

It is easy to see that the outer sum and the outer product are commutative. However, for $n, m \in \mathbb{N} \subseteq \mathit{Card}$, $n+m$ and $n \cdot m$ are generally not the same objects as $n \oplus m$ and $n \odot m$ respectively.

Theorem 4.5. *Let X be a set and $A \subseteq X$. Then $|X \setminus A| \oplus |A| = T^2(|X|)$.*

Proof: We need to prove $(X \setminus A) \times \{0\} \cup A \times \{1\} \sim \mathscr{P}_1^2(X)$. Since $\iota^2(x)$ is type-level with $(x, 0)$ and $\iota^2(y)$ is type-level with $(y, 1)$, by corollary 3.13 we can define functions $h_1(x, 0) := \iota^2(x), x \in X \setminus A$ and $h_2(y, 1) := \iota^2(y), y \in A$. Then obviously $\mathit{bij}\big(h_1 \cup h_2, (X \setminus A) \times \{0\} \cup A \times \{1\}, \mathscr{P}_1^2(X)\big)$. □

Theorem 4.6. *For every infinite cardinal number κ and for every natural number n we have $\kappa \oplus n = T^2(\kappa)$.*

Proof: Let $\kappa = |X|$ and $n = |A|$. We need to prove $X \times \{0\} \cup A \times \{1\} \sim \mathscr{P}_1^2(X)$. If $n = 0$, then $A = \emptyset = A \times \{1\}$. Therefore, we need to prove $X \times \{0\} \sim \mathscr{P}_1^2(X)$, and one bijection is $(x, 0) \mapsto \iota^2(x)$.

Let $n \neq 0$. Then from theorem 3.20 there exists an injection $f\colon \mathbb{N} \to X$ and by theorem 3.16(6) there exists an injection $g\colon A \to \mathbb{N}$. By assumption, A is a finite set, therefore $\mathit{rng}(g)$ is finite (and nonempty). From theorem 3.16(5) it follows that $\mathit{rng}(g)$ has the greatest element a_0.

By application of corollary 3.13, we can define the following functions:

$$h_1(x,0) := \iota^2\big(f\big(f^{-1}(x) + a_0 + 1\big)\big),\ x \in \mathit{rng}(f)$$
$$h_2(x,0) := \iota^2(x),\ x \in X \setminus \mathit{rng}(f)$$
$$h_3(a,1) := \iota^2\big(f(g(a))\big),\ a \in A.$$

Then $\mathit{bij}(h_1 \cup h_2 \cup h_3, X \times \{0\} \cup A \times \{1\}, \mathscr{P}_1^2(X))$ can be proved by cases, and that means we have $\kappa \oplus n = T^2(\kappa)$. $\square$

THEOREM 4.7. $\aleph_0 \oplus \aleph_0 = T^2(\aleph_0)$.

PROOF: We need to prove $\mathbb{N} \times \{0\} \cup \mathbb{N} \times \{1\} \sim \mathscr{P}_1^2(\mathbb{N})$. By corollary 3.13 we can define functions $f_1(n,0) := \iota^2(2 \cdot n), n \in \mathbb{N}$ and $f_2(n,1) := \iota^2(2 \cdot n + 1), n \in \mathbb{N}$. Obviously, $\mathit{bij}\big(f_1 \cup f_2, \mathbb{N} \times \{0\} \cup \mathbb{N} \times \{1\}, \mathscr{P}_1^2(\mathbb{N})\big)$. $\square$

THEOREM 4.8. *For every infinite cardinal κ we have $\kappa \oplus \kappa = T^2(\kappa)$.*

PROOF: Let $\kappa = |X|$. We need to prove $X \times \{0\} \cup X \times \{1\} \sim \mathscr{P}_1^2(X)$. Formula $\exists Y^2\big(Y^2 \subseteq X^2 \wedge Y^2 \notin \mathit{FIN}^3 \wedge \mathit{bij}(f^6, (Y^2 \times \{0^1\}^2)^4 \cup (Y^2 \times \{1^1\}^2)^4, \mathscr{P}_1^2(Y^2)^4)\big)$ is stratified so we define the set $K := \big\{f \mid \exists Y\big(Y \subseteq X \wedge Y \notin \mathit{FIN} \wedge \mathit{bij}(f, Y \times \{0\} \cup Y \times \{1\}, \mathscr{P}_1^2(Y))\big)\big\}$, which we order by inclusion. Because X is infinite, by theorem 3.20 there exists a countable $X_0 \subseteq X$. Now from theorem 4.7 we get $X_0 \times \{0\} \cup X_0 \times \{1\} \sim \mathscr{P}_1^2(X_0)$, so there exists a bijection $f_0 \colon X_0 \times \{0\} \cup X_0 \times \mathscr{P}_1^2(X_0)$, which means $f_0 \in K$, so K is nonempty. Let C be an arbitrary nonempty chain in K. By lemma 2.13 we get that $\bigcup C$ is an injection. We need to prove $\bigcup C \in K$.

Formula $(\exists f^5 \in C^6)\big((\bigcup\bigcup \mathit{rng}(f^5)^3)^1 = z^1\big)$ is stratified, so we define the set $S := \big\{z \mid (\exists f \in C)\big(\bigcup\bigcup \mathit{rng}(f) = z\big)\big\}$. Application of the double union on the set $\mathit{rng}(f)$ is necessary to get rid of double $\mathscr{P}_1$. We claim $\bigcup S \subseteq X$, $\bigcup S$ is infinite, and $\mathit{rng}\big(\bigcup C\big) = \mathscr{P}_1^2\big(\bigcup S\big)$.

Let us prove $\bigcup S \subseteq X$. Let $z \in \bigcup S$. Then there exists $t \in S$ such that $z \in t$. There exists $f \in C$ such that $t = \bigcup\bigcup \mathit{rng}(f)$ and $z \in t$. From $\mathit{rng}(f) \subseteq \mathscr{P}_1^2(X)$, we have $\mathit{rng}(f) = \mathscr{P}_1^2(A)$ for some infinite $A \subseteq X$. But then $\mathscr{P}_1^2(t) = \mathscr{P}_1^2\big(\bigcup\bigcup \mathit{rng}(f)\big) = \mathscr{P}_1^2\big(\bigcup\bigcup \mathscr{P}_1^2(A)\big) = \mathscr{P}_1^2(A) = \mathit{rng}(f)$. Therefore, $t \subseteq X$, which implies $z \in X$.

Let us prove that $\bigcup S$ is infinite. Assume the contrary, that it is finite. Let us fix $f \in C$. Then $\mathit{rng}(f) = \mathscr{P}_1^2(A)$ for some infinite $A \subseteq X$. Now we have $A = \bigcup\bigcup \mathscr{P}_1^2(A) = \bigcup\bigcup \mathit{rng}(f) \subseteq \bigcup S$. From lemma 3.15 we get that A is finite, which is a contradiction.

Let us prove $rng(\bigcup C) = \mathscr{P}_1^2(\bigcup S)$. If $z \in rng(\bigcup C)$, then by lemma 2.13 there exists $f \in C$ such that $z \in rng(f)$. That means that there exists infinite $A \subseteq X$ such that $z \in rng(f) = \mathscr{P}_1^2(A)$. Then there exists $a \in A$ such that $z = \iota^2(a)$. We have $\bigcup\bigcup \iota^2(a) = a \in A = \bigcup\bigcup \mathscr{P}_1^2(A) = \bigcup\bigcup rng(f)$, so $A \in S$, and from that we get $a \in \bigcup S$. Now we have $z = \iota^2(a) \in \mathscr{P}_1^2(\bigcup S)$.

If $z \in \mathscr{P}_1^2(\bigcup S)$, then there exists $b \in \bigcup S$ such that $z = \iota^2(b)$. That means there exists $B \in S$ such that $b \in B$. That implies there exists $f \in C$ such that $\bigcup\bigcup rng(f) = B$ and $b \in B$. We know that $rng(f) = \mathscr{P}_1^2(A)$ for some infinite $A \subseteq X$. From that we get $z = \iota^2(b) \in \mathscr{P}_1^2(B) = \mathscr{P}_1^2(\bigcup\bigcup rng(f)) = \mathscr{P}_1^2(\bigcup\bigcup \mathscr{P}_1^2(A)) = \mathscr{P}_1^2(A) = rng(f) \subseteq rng(\bigcup C)$. So, there exists infinite $Z := \bigcup S \subseteq X$ such that $rng(\bigcup C) = \mathscr{P}_1^2(Z)$.

It remains to prove $dom(\bigcup C) = Z \times \{0\} \cup Z \times \{1\}$. Let $z \in dom(\bigcup C)$, then by lemma 2.13 there exists $f \in C$ such that $z \in dom(f) = T \times \{0\} \cup T \times \{1\}$, for some infinite $T \subseteq X$. Because $f \subseteq \bigcup C$, we have $\mathscr{P}_1^2(T) = rng(f) \subseteq rng(\bigcup C) = \mathscr{P}_1^2(Z)$, which implies $T \subseteq Z$, that is, $T \times \{0\} \cup T \times \{1\} \subseteq Z \times \{0\} \cup Z \times \{1\}$. Therefore, $z \in Z \times \{0\} \cup Z \times \{1\}$, that is, $dom(\bigcup C) \subseteq Z \times \{0\} \cup Z \times \{1\}$.

If $z \in Z \times \{0\} \cup Z \times \{1\}$, then $z = (a, s)$, where $a \in Z$ and $s \in \{0, 1\}$. Then $\iota^2(a) \in \mathscr{P}_1^2(Z) = rng(\bigcup C)$, which means there exists $f \in C$ such that $\iota^2(a) \in rng(f) = \mathscr{P}_1^2(U)$, for infinite $U \subseteq X$. Then we have $a \in U$, which implies $z = (a, s) \in U \times \{0\} \cup U \times \{1\} = dom(f)$, so $z \in dom(\bigcup C)$.

Finally, we can conclude $\bigcup C \in K$, and then by Zorn's lemma, there exists a maximal element of K. Denote it by $f_0 \colon A_0 \times \{0\} \cup A_0 \times \{1\} \to \mathscr{P}_1^2(A_0)$, where $A_0 \subseteq X$ is infinite. We want to prove $|X| = |A_0|$.

By theorem 4.5 we have $T^2(|X|) = |X \setminus A_0| \oplus |A_0|$. We claim that $X \setminus A_0$ is finite. Assume the contrary, that it is infinite. Then there exists a countable set $B \subseteq X \setminus A_0$, which implies $A_0 \subsetneq A_0 \cup B \subseteq X$. Because B is countable, by theorem 4.7 we get $B \times \{0\} \cup B \times \{1\} \sim \mathscr{P}_1^2(B)$, so there exists a bijection $g_0 \colon B \times \{0\} \cup B \times \{1\} \to \mathscr{P}_1^2(B)$. Obviously, $bij(f_0 \cup g_0, (A_0 \cup B) \times \{0\} \cup (A_0 \cup B) \times \{1\}, \mathscr{P}_1^2(A_0 \cup B))$. But now we have $f_0 \subset f_0 \cup g_0$, which is a contradiction with the maximality of f_0. Therefore, $X \setminus A_0$ is finite. Now from theorem 4.6 we get $T^2(|X|) = |X \setminus A_0| \oplus |A_0| = T^2(|A_0|)$, that is, $|X| = |A_0|$. $\square$

Theorem 4.9. *Let κ be an infinite cardinal and $\lambda \leq \kappa$. Then $\kappa \oplus \lambda = \lambda \oplus \kappa = T^2(\kappa)$.*

Proof: Let $A \in \kappa$ and $B \in \lambda$. We need to prove $A \times \{0\} \cup B \times \{1\} \sim \mathscr{P}_1^2(A)$. From $\lambda \leq \kappa$ we have an injection $f\colon B \to A$. Denote $X := \mathit{rng}(f)$, which is obviously equipotent with B. By theorem 4.8 we have $\mathscr{P}_1^2(X) \sim X \times \{0\} \cup X \times \{1\} \sim X \times \{0\} \cup B \times \{1\}$. Then $A \times \{0\} \cup B \times \{1\} = (A \setminus X \cup X) \times \{0\} \cup B \times \{1\} = (A \setminus X) \times \{0\} \cup (X \times \{0\} \cup B \times \{1\}) \sim \mathscr{P}_1^2(A \setminus X) \cup \mathscr{P}_1^2(X) \sim \mathscr{P}_1^2(A \setminus X \cup X) = \mathscr{P}_1^2(A)$. □

Lemma 4.10. *For any family of finitely many equipotent infinite sets, their union is also equipotent with each of them.*

Proof: Denote the number of sets with n. The claim is trivial for $n = 0$ and $n = 1$. It is enough to prove the claim for $n = 2$; then the claim for $n \geq 3$ follows by induction.

Let $A \sim B$ be arbitrary sets and define $C := A \setminus B$. Then $C \subseteq A$, which means $\mathit{inj}(\mathit{id}_C, C, A)$, so $|C| \leq |A|$. By theorem 4.9, $|A| \oplus |C| = T^2(|A|)$. On the other hand, for $U := A \cup B = C \cup B$ we have $U \setminus C = B$, so by theorem 4.5, we have $|A| \oplus |C| = |B| \oplus |C| = |U \setminus C| \oplus |C| = T^2(|U|)$. From these two facts, $|A| = |U|$ follows. □

Theorem 4.11. $\aleph_0 \odot \aleph_0 = T^2(\aleph_0)$.

Proof: Formula $(\exists n^1 \in \mathbb{N}^2)\big(a^3 = \iota^2(n^1)^3 \wedge b^3 = (n^1, n^1)^3 \wedge t^5 = (a^3, b^3)^5\big)$ is stratified, so we can define a relation $g := \big\{\big(\iota^2(n), (n, n)\big) \bigm| n \in \mathbb{N}\big\}$. Then $\mathit{inj}\big(g, \mathscr{P}_1^2(\mathbb{N}), \mathbb{N} \times \mathbb{N}\big)$, which implies $T^2(\aleph_0) \leq \aleph_0 \odot \aleph_0$.

By corollary 3.13 we can define a function $f(m, n) := \iota^2\big((m+n) \cdot (m+n) + m\big)$ for every $m, n \in \mathbb{N}$. We need to prove that f is an injection. Let $n, m, a, b \in \mathbb{N}$ be such that $(m, n) \neq (a, b)$.

The first case is when $m + n \neq a + b$, without the loss of generality $m + n < a + b$. Then $m + n + 1 = \mathit{succ}(m + n) \leq a + b$. So we have

$$\begin{aligned}(m+n) \cdot (m+n) + m &\leq (m+n) \cdot (m+n) + m + n + m + n < \\ &< \mathit{succ}\big((m+n) \cdot (m+n+2)\big) = (m+n+1) \cdot (m+n+1) \leq \\ &\leq (a+b) \cdot (m+n+1) \leq (a+b) \cdot (a+b) \leq (a+b) \cdot (a+b) + a,\end{aligned}$$

which implies $f(m, n) \neq f(a, b)$.

The second case is when $m + n = a + b$, and then obviously $m \neq a$, without the loss of generality $m < a$. Then we have $(m+n) \cdot (m+n)+m = (a + b) \cdot (a + b) + m < (a + b) + a$, and also $f(m, n) \neq f(a, b)$.

Therefore, $f \colon \mathbb{N} \times \mathbb{N} \to \mathscr{P}^2_1(\mathbb{N})$ is an injection, so we have $\aleph_0 \times \aleph_0 \leq T^2(\aleph_0)$. By Cantor–Bernstein's theorem we get $\aleph_0 \times \aleph_0 = T^2(\aleph_0)$. □

Theorem 4.12. *For every infinite cardinal κ we have $\kappa \odot \kappa = T^2(\kappa)$.*

Proof: Let $\kappa = |X|$. We need to prove $X \times X \sim \mathscr{P}^2_1(X)$. Formula $\exists Y^1\big(Y^1 \subseteq X^1 \wedge Y^1 \notin \mathit{FIN}^2 \wedge \mathit{bij}(f^5, (Y \times Y)^3), \mathscr{P}^2_1(Y)^3\big)$ is stratified, so we can define the set $K := \big\{f \bigm| \exists Y\big(Y \subseteq X \wedge Y \notin \mathit{FIN} \wedge \mathit{bij}\big(f, Y \times Y, \mathscr{P}^2_1(Y)\big)\big)\big\}$, which we order by inclusion. By theorem 3.20 there exists a countable $X_0 \subseteq X$ and by theorem 4.11 we have $X_0 \times X_0 \sim \mathscr{P}^2_1(X_0)$, so there exists a bijection $f_0 \colon X_0 \times X_0 \to \mathscr{P}^2_1(X_0)$, which means $f_0 \in K$, so K is nonempty. Let C be an arbitrary nonempty chain in K. By lemma 2.13 we get that $\bigcup C$ is an injection. We need to prove $\bigcup C \in K$.

We can prove analogously as in the proof of theorem 4.8 that there exists an infinite $Z \subseteq X$ such that $\mathit{rng}\big(\bigcup C\big) = \mathscr{P}^2_1(Z)$.

It remains to prove $\mathit{dom}\big(\bigcup C\big) = Z \times Z$. Let $z \in \mathit{dom}\big(\bigcup C\big)$, then by lemma 2.13 there exists $f \in C$ such that $z \in \mathit{dom}(f) = T \times T$, for some infinite $T \subseteq X$. Because $f \subseteq \bigcup C$, we have $\mathscr{P}^2_1(T) = \mathit{rng}(f) \subseteq \mathit{rng}\big(\bigcup C\big) = \mathscr{P}^2_1(Z)$, which implies $T \subseteq Z$, that is, $T \times T \subseteq Z \times Z$. Therefore, $z \in Z \times Z$ and then we have $\mathit{dom}\big(\bigcup C\big) \subseteq Z \times Z$.

If $z \in Z \times Z$, then $z = (u, w)$, where $u, w \in Z$. Then $\iota^2(u), \iota^2(w) \in \mathscr{P}^2_1(Z) = \mathit{rng}\big(\bigcup C\big)$, which means there exist $f_1, f_2 \in C$ such that $\iota^2(u) \in \mathit{rng}(f_1)$ and $\iota^2(w) \in \mathit{rng}(f_2)$. Because C is a chain, without the loss of generality we can assume $f_1 \subseteq f_2$, therefore, $\iota^2(u), \iota^2(w) \in \mathit{rng}(f_2) = \mathscr{P}^2_1(U)$ for some infinite $U \subseteq X$. Then we have $u, w \in U$, which implies $z = (u, w) \in U \times U = \mathit{dom}(f_2) \subseteq \mathit{dom}\big(\bigcup C\big)$, so $z \in \mathit{dom}\big(\bigcup C\big)$.

We can conclude $\bigcup C \in K$, and then by Zorn's lemma, there exists a maximal element of K. Denote it by $f_0 \colon A_0 \times A_0 \to \mathscr{P}^2_1(A_0)$, where $A_0 \subseteq X$ is infinite. Then f_0 shows $\lambda \odot \lambda = T^2(\lambda)$, where $\lambda := |A_0|$. It remains to prove $|A_0| = |X|$.

From $A_0 \subseteq X$, we get $|A_0| \leq |X|$. Assume $|A_0| < |X|$. Because $\leq$ is well-order, either $|X \setminus A_0| \leq |A_0|$ or $|A_0| < |X \setminus A_0|$. If $|X \setminus A_0| \leq |A_0|$, by theorems 4.5 and 4.9 we have $T^2(|X|) = |X \setminus A_0| \oplus |A_0| = T^2(|A_0|)$, so we get $|X| = |A_0|$, a contradiction. Therefore, $|A_0| < |X \setminus A_0|$, so there exists an injection from A_0 to $X \setminus A_0$, which is not a bijection; hence there exists $Z \subset X \setminus A_0$ such that $|Z| = |A_0| = \lambda$.

By distributivity we have $(A_0 \cup Z) \times (A_0 \cup Z) = (A_0 \times A_0) \cup (A_0 \times Z) \cup (Z \times A_0) \cup (Z \times Z)$. Now from $A_0 \sim Z$ we get $A_0 \times Z \sim Z \times A_0 \sim Z \times Z$, and then from lemma 4.10

$$(A_0 \times Z) \cup (Z \times A_0) \cup (Z \times Z) \sim Z \times Z \sim \mathscr{P}_1^2(Z).$$

Therefore, there exists a bijection $g\colon (A_0 \times Z) \cup (Z \times A_0) \cup (Z \times Z) \to \mathscr{P}_1^2(Z)$.

Define $h := (f_0 \cup g) : (A_0 \cup Z) \times (A_0 \cup Z) \to \mathscr{P}_1^2(A_0 \cup Z)$. Since $A_0 \cap Z = \emptyset$, h is a bijection such that $f_0 \subseteq h$, and $h \in K$, because $A_0 \cup Z$ is an infinite subset of X.

We also have $f_0 \neq h$ because for any $z \in Z \neq \emptyset$, $\big((z,z), \iota^2(z)\big) \in h \setminus f_0$, since $Z \subseteq X \setminus A_0$. Now we have $f_0 \subset h \in K$, a contradiction with the maximality of f_0.

Therefore, the assumption $|A_0| < |X|$ was wrong, which implies $|X| \leq |A_0|$, so $\lambda = |A_0| = |X| = \kappa$. Now $\kappa \odot \kappa = \lambda \odot \lambda = T^2(\lambda) = T^2(\kappa)$. $\square$

Remark 4.13. The proofs of theorems about cardinal arithmetic are good examples of why working with Kuratowski's ordered pair (or any other pairs that are not type-level) is tedious. Even the statements of theorems must be modified in order to accommodate this. Using type-level ordered pairs greatly reduces the complexity of said proofs.

THEOREM 4.14. *In* $\mathsf{NFU} + \mathsf{Inf} + \mathsf{AC}$ *there exist type-level ordered pairs.*

PROOF: Denote the cardinal number of the universe as $|V| =: \kappa$. We know from the axiom of infinity that V is an infinite set, so κ is an infinite cardinal number. From theorem 4.12 we have $\kappa \odot \kappa = T^2(\kappa)$, which means there is a bijection $F\colon V \times V \to \mathscr{P}_1^2(V)$.

Formula $F^6\big((x^1, y^1)^3\big)^3 = \iota^2(w^1)^3$ is stratified, so we can define the set $S_{xy} := \{w \mid F\big((x,y)\big) = \iota^2(w)\}$. Note that S_{xy} is a singleton: for if $z_1, z_2 \in S_{xy}$, then $\iota^2(z_1) = F\big((x,y)\big) = \iota^2(z_2)$, which implies $z_1 = z_2$.

For $x, y \in V$ we define new ordered pair $\langle x, y\rangle := \bigcup S_{xy}$. Let us prove that it satisfies the usual property of ordered pairs and that it is type-level.

Let us first prove the usual property. Let $x, y, a, b \in V$ be such that $\langle x, y\rangle = \langle a, b\rangle$. By definition, we have $F\big((x,y)\big) = \iota^2(\langle x,y\rangle)$ and $F\big((a,b)\big) = \iota^2(\langle a,b\rangle)$, so $F\big((x,y)\big) = F\big((a,b)\big)$. Since F is an injection, we have $(x,y) = (a,b)$, which implies $x = a$ and $y = b$. If $x = a$ and $y = b$, then $\iota^2(\langle x,y\rangle) = F\big((x,y)\big) = F\big((a,b)\big) = \iota^2(\langle a,b\rangle)$, which implies $\langle x,y\rangle = \langle a,b\rangle$.

Let us prove that they are type-level. Let $x, y \in V$ be arbitrary. We have

$$z^1 \in \langle x,y\rangle^2 \leftrightarrow \exists w^2\big(F^7\big((x^2,y^2)^4\big)^4 = \iota^2(w^2)^4 \land z^1 \in w^2\big).$$

That proves that if x and y have type n, then $\langle x,y\rangle$ has type n. Therefore, we have type-level ordered pairs. □

Remark 4.15. Since here we're primarily concerned with set theory and not with logic, we are somewhat sloppy with respect to proving existence versus "pinpointing" some mathematical object. However, in the interest of completeness, it is important to note that using the logical principle of *existential instantiation* [3] we can in fact, having proved $\exists F\, \mathit{bij}\big(F, V \times V, \mathscr{P}_1^2(V)\big)$, expand the signature of our theory by a new constant symbol F and an axiom $\mathit{bij}\big(F, V \times V, \mathscr{P}_1^2(V)\big)$, and it will be a conservative extension. Then the new constant symbol can be used in other ways, for instance, to define a two-place function term for the new ordered pair $\langle \_ , \_ \rangle$.

5. Axiomatic extension

We will briefly show how to use the third approach from the introduction. We start by introducing axioms of NFU (the axiom of extensionality, the axiom of sethood, and the axiom of stratified comprehension). Next, we need a few basic notions independent of the usage of ordered pairs.

We are then able to introduce the axiom of choice. The next step is to introduce natural numbers or, more precisely, the notion of finite sets. Then we are able to introduce the axiom of infinity.

The only thing left is the introduction of the notion of (Kuratowski's) bijection and then we can state the *universe cardinal squaring principle*.

DEFINITION 5.1.

1. For $x, y \in V$ we define their **Kuratowski's ordered pair** $(x,y)_K := \big\{\{x\},\{x,y\}\big\}$.
2. For $X, Y \in \mathit{SET}$ we define their **Kuratowski's product** $X \times_K Y := \{(x,y)_K \mid x \in X \land y \in Y\}$.
3. For X and Y we define the notion of **Kuratowski's bijection** between them as $\mathit{bij}_K(f,X,Y) :\Leftrightarrow f \subseteq X \times_K Y \land \forall x \exists! y\big((x,y)_K \in f\big) \land \land \forall y \exists! x\big((x,y)_K \in f\big)$.

Universe cardinal squaring principle:

$$V \notin FIN \to \exists f\ bij_K\big(f, V \times_K V, \mathscr{P}_1^2(V)\big).$$

The *universe cardinal squaring principle* can be interpreted as a claim that there exists a (Kuratowski's) bijection between $V \times_K V$ and $\mathscr{P}_1^2(V)$.

The finishing touch is theorem 4.14, and via it, we can define (see remark 4.15) type-level ordered pairs. We can now develop the theory in any way needed. Notions independent of ordered pairs will stay the same, and few should be redefined, replacing Kuratowski's definitions with type-level ones.

One important notion that should also be redefined is the notion of applying the function to an argument, since now the type of f must be only one higher than the type of x, in order for $f(x)$ to be meaningful and have a type (equal to the type of x).

The next two results were given to us by an anonymous reviewer.

Theorem 5.2. $\mathsf{NFU} + \mathsf{OP}$ *proves* VCSP.

Proof: Denote with $(x, y)_K$ Kuratowski's ordered pairs and with (x, y) type-level ordered pairs. Assume $V \notin FIN$. Since $(x, y)_K$ and $\iota^2\big((x, y)\big)$ have the same type, by corollary 3.13 we can define a function $f\big((x, y)_K\big) = \iota^2\big((x, y)\big)$ for every $x, y \in V$. Function f is obviously an injection from $V \times_K V$ to $\mathscr{P}_1^2(V)$. On the other hand, function $\iota^2(x) \mapsto (x, x)_K$ is obviously an injection from $\mathscr{P}_1^2(V) \to V \times_K V$. Now Cantor–Bernstein's theorem implies that there exists a bijection between $V \times_K V$ and $\mathscr{P}_1^2(V)$. Therefore, the universe cardinal squaring principle holds. □

Theorem 5.3. $\mathsf{NFU} + \mathsf{Inf} + \mathsf{VCSP}$ *does not prove* AC.

Proof (Sketch of proof:): First, we know that $\mathsf{NFU} + \mathsf{Inf}$ interprets $\mathsf{NFU} + \mathsf{OP}$: within any model M of $\mathsf{NFU} + \mathsf{Inf}$ we can find a smaller model M' of $\mathsf{NFU} + \mathsf{OP}$. More precisely, M' is obtained as a doubly iterated partitive set of V in M [5]. Therefore, the truth of Zermelo's theorem (and also of AC) is the same in both M and M'.

We also know that $\mathsf{NFU}+\mathsf{Inf}$ does not prove AC [11]: there is a model M of $\mathsf{NFU} + \mathsf{Inf}$ which does not validate AC. If we carry out the construction from the previous paragraph, we obtain M' which validates NFU and also Inf [9], while proving VCSP by theorem 5.2 and not validating AC (since if AC were to hold in M', it would also hold in M by the previous paragraph, which is a contradiction). □

Conclusion

It is apparent that non-type-level ordered pairs are causing many difficulties. By proving the cardinal squaring principle using Kuratowski's ordered pairs we are able to justify $\mathsf{NFU}+\mathsf{Inf}+\mathsf{AC}+\mathsf{VCSP}$. Not only that, we have presented the development of NFU with Kuratowski's ordered pairs that can be used for further reference, without the need to go through it again every time type-level ordered pairs are needed.

It is worth emphasizing that everything in this article is done without the appeal to Rosser's axiom of counting, which is prominently used in Rosser's [13] and Holmes' book [7]. In our opinion, this shows that the usage of the axiom of counting, although sometimes making proofs simpler, is not essential to our approach.

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