Bulletin of the **S**ection of **L**ogic
Volume 45/1 (2016), pp. 53–63

http://dx.doi.org/10.18778/0138-0680.45.1.04

Young Bae Jun, Eun Hwan Roh and Seok Zun Song*

COMMUTATIVE ENERGETIC SUBSETS OF BCK-ALGEBRAS

Abstract

The notions of a C-energetic subset and (anti) permeable C-value in BCK-algebras are introduced, and related properties are investigated. Conditions for an element t in $[0, 1]$ to be an (anti) permeable C-value are provided. Also conditions for a subset to be a C-energetic subset are discussed. We decompose BCK-algebra by a partition which consists of a C-energetic subset and a commutative ideal.

Keywords: S-energetic subset, I-energetic subset, C-energetic subset, (anti) fuzzy commutative ideal, (anti) permeable I-value, (anti) permeable C-value.

2010 Mathematics Subject Classification. 06F35, 03G25, 08A72.

1. Introduction

Jun et al. [3] introduced the notions of energetic (resp. right vanished, right stable) subsets and (anti) permeable values in BCK/BCI-algebras. Using the notion of (anti) fuzzy subalgebras/ideals of BCK/BCI-algebras, they investigated relations among subalgebras/ideals, energetic subsets, (anti) permeable values, right vanished subsets and right stable subsets.

*Corresponding author.

In this article, we introduce the notions of a C-energetic subset and (anti) permeable C-value in BCK-algebras, and investigate related properties. We provide conditions for an element t in $[0,1]$ to be an (anti) permeable C-value. We also discuss conditions for a subset to be a C-energetic subset. We show that a BCK-algebra is decomposed by a partition which consists of a C-energetic subset and a commutative ideal.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* it satisfies the following conditions

(I) $(\forall x,y,z\in X)\ (((x*y)*(x*z))*(z*y)=0)$,
(II) $(\forall x,y\in X)\ ((x*(x*y))*y=0)$,
(III) $(\forall x\in X)\ (x*x=0)$,
(IV) $(\forall x,y\in X)\ (x*y=0, y*x=0\ \Rightarrow\ x=y)$.

If a BCI-algebra X satisfies the following identity

(V) $(\forall x\in X)\ (0*x=0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following axioms

$$(\forall x\in X)\,(x*0=x)\,, \tag{2.1}$$
$$(\forall x,y,z\in X)\,(x\le y\ \Rightarrow\ x*z\le y*z,\ z*y\le z*x)\,, \tag{2.2}$$
$$(\forall x,y,z\in X)\,((x*y)*z=(x*z)*y)\,, \tag{2.3}$$
$$(\forall x,y,z\in X)\,((x*z)*(y*z)\le x*y) \tag{2.4}$$

where $x\le y$ if and only if $x*y=0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x*y\in S$ for all $x,y\in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies

$$0\in I, \tag{2.5}$$
$$(\forall x\in X)\,(\forall y\in I)\,(x*y\in I\ \Rightarrow\ x\in I)\,. \tag{2.6}$$

A subset I of a BCK-algebra X is called a *commutative ideal* (see [5]) of X if it satisfies (2.5) and

$$(\forall x,y\in X)(\forall z\in I)\,((x*y)*z\in I\ \Rightarrow\ x*(y*(y*x))\in I)\,. \tag{2.7}$$

Observe that every commutative ideal is an ideal, but the converse is not true (see [6]).

We refer the reader to the books [2, 6] for further information regarding BCK/BCI-algebras.

The concept of fuzzy sets was introduced by Zadeh [7]. Let X be a set. The mapping $f : X \to [0, 1]$ is called a *fuzzy set* in X.

A fuzzy set f in a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X if it satisfies

$$(\forall x, y \in X)\,(f(x * y) \geq \min\{f(x), f(y)\})\,. \tag{2.8}$$

A fuzzy set f in a BCK/BCI-algebra X is called a *fuzzy ideal* of X if it satisfies

$$(\forall x \in X)\,(f(0) \geq f(x))\,. \tag{2.9}$$

$$(\forall x, y \in X)\,(f(x) \geq \min\{f(x * y), f(y)\})\,. \tag{2.10}$$

Note that every fuzzy ideal f of a BCK/BCI-algebra X satisfies

$$(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; f(x) \geq f(y))\,. \tag{2.11}$$

A fuzzy set f in a BCK-algebra X is called a *fuzzy commutative ideal* (see [4]) of X if it satisfies (2.9) and

$$(\forall x, y, z \in X)\,(f(x * (y * (y * x))) \geq \min\{f((x * y) * z), f(z)\})\,. \tag{2.12}$$

For a fuzzy set f in X and $t \in [0, 1]$, the (strong) upper (resp. lower) t-level sets are defined as follows:

$$U(f; t) := \{x \in X \mid f(x) \geq t\},\;\; U^*(f; t) := \{x \in X \mid f(x) > t\},$$
$$L(f; t) := \{x \in X \mid f(x) \leq t\},\;\; L^*(f; t) := \{x \in X \mid f(x) < t\}.$$

3. Commutative energetic subsets

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 3.1 ([3]). *A non-empty subset A of X is said to be S-energetic if it satisfies*

$$(\forall a, b \in X)\,(a * b \in A \;\Rightarrow\; \{a, b\} \cap A \neq \emptyset)\,. \tag{3.1}$$

Definition 3.2 ([3]). *A non-empty subset A of X is said to be I-energetic if it satisfies*

$$(\forall x, y \in X)\,(y \in A \;\Rightarrow\; \{x, y * x\} \cap A \neq \emptyset)\,. \tag{3.2}$$

Lemma 3.3 ([3]). *For any subset A of X, if $X \setminus A$ is an ideal of X, then A is I-energetic.*

Definition 3.4. *A non-empty subset A of X is said to be commutative energetic (briefly, C-energetic) if it satisfies*

$$(\forall x, y, z \in X)\,(x * (y * (y * x)) \in A \;\Rightarrow\; \{z, (x * y) * z\} \cap A \neq \emptyset)\,. \quad (3.3)$$

Example 3.5. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

It is routine to verify that $A := \{3, 4\}$ is a C-energetic subset of X.

We consider relations between an I-energetic subset and a C-energetic subset.

Theorem 3.6. *Every C-energetic subset is I-energetic.*

Proof: Let A be a C-energetic subset of X. Let $x, y \in X$ be such that $y \in A$. Then $y * (0 * (0 * y)) = y \in A$, and so $\{x, (y * 0) * x\} \cap A \neq \emptyset$ by (3.3). It follows from (2.1) that $\{x, y * x\} \cap A \neq \emptyset$. Hence A is an I-energetic subset of X. □

The converse of Theorem 3.6 is not true as seen in the following examples.

Example 3.7. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Take $A := \{1, 2, 4\}$. Then $X \setminus A = \{0, 3\}$ is an ideal of X. Hence, by Lemma 3.3, A is an I-energetic subset of X. But it is not C-energetic since

$$1 * (4 * (4 * 1)) = 1 \in A \quad \textit{and} \quad \{3,\ (1 * 4) * 3\} \cap A = \emptyset.$$

THEOREM 3.8. *For any nonempty subset A of X, if $X \setminus A$ is a commutative ideal of X, then A is C-energetic.*

PROOF: Assume that A is not C-energetic. Then for any $x, y \in X$ with
$$x * (y * (y * x)) \in A,$$
there exists $z \in X$ such that $\{z,\ (x * y) * z\} \cap A = \emptyset$. It follows that
$$(x * y) * z \in X \setminus A \text{ and } z \in X \setminus A.$$
Since $X \setminus A$ is a commutative ideal of X, we have $x * (y * (y * x)) \in X \setminus A$, that is, $x*(y*(y*x)) \notin A$. This is a contradiction, and so A is a C-energetic subset of X. □

COROLLARY 3.9. *For any nonempty subset A of X, if $X \setminus A$ is a commutative ideal of X, then A is I-energetic.*

THEOREM 3.10. *Let A be a nonempty subset of X with $0 \notin A$. If A is C-energetic, then $X \setminus A$ is a commutative ideal of X.*

PROOF: Obviously $0 \in X \setminus A$. Let $x, y, z \in X$ be such that $z \in X \setminus A$ and $(x*y)*z \in X\setminus A$. Assume that $x*(y*(y*x)) \in A$. Then $\{z,\ (x*y)*z\} \cap A \neq \emptyset$ by (3.3), which implies that $z \in A$ or $(x*y)*z \in A$. This is a contradiction, and so $x * (y * (y * x)) \in X \setminus A$. This shows that $X \setminus A$ is a commutative ideal of X. □

COROLLARY 3.11. *Let A be a nonempty subset of X with $0 \notin A$. If A is C-energetic, then $X \setminus A$ is an ideal and hence a subalgebra of X.*

THEOREM 3.12. *If f is a fuzzy commutative ideal of X, then the nonempty lower t-level set $L(f;t)$ is a C-energetic subset of X.*

PROOF: Assume that $L(f;t) \neq \emptyset$ for $t \in [0,1]$. Let $x, y \in X$ be such that $x * (y * (y * x)) \in L(f;t)$. Then

$$t \geq f(x * (y * (y * x))) \geq \min\{f((x * y) * z), f(z)\}$$

for all $z \in X$, which implies that $f((x * y) * z) \leq t$ or $f(z) \leq t$, that is, $(x * y) * z \in L(f;t)$ or $z \in L(f;t)$. Thus $\{z, (x * y) * z\} \cap L(f;t) \neq \emptyset$, and therefore $L(f;t)$ is a C-energetic subset of X. □

COROLLARY 3.13. *If f is a fuzzy commutative ideal of X, then the nonempty strong lower t-level set $L^*(f;t)$ is a C-energetic subset of X.*

Since $L(f;t) \cup U^*(f;t) = X$ and $L(f;t) \cap U^*(f;t) = \emptyset$ for all $t \in [0,1]$, we have the following corollary.

COROLLARY 3.14. *If f is a fuzzy commutative ideal of X, then $U^*(f;t)$ is empty or a commutative ideal of X for all $t \in [0,1]$.*

DEFINITION 3.15 ([1]). *A fuzzy set f in X is called an anti fuzzy ideal of X if it satisfies*

$$(\forall x \in X)\,(f(0) \leq f(x))\,. \tag{3.4}$$

$$(\forall x, y \in X)\,(f(x) \leq \max\{f(x*y), f(y)\})\,. \tag{3.5}$$

DEFINITION 3.16. *A fuzzy set f in X is called an anti fuzzy commutative ideal of X if it satisfies* (3.4) *and*

$$(\forall x, y, z \in X)\,(f(x*(y*(y*x))) \leq \max\{f((x*y)*z), f(z)\})\,. \tag{3.6}$$

EXAMPLE 3.17. *Consider a BCK-algebra $X = \{0, a, b, c\}$ with the following Cayley table*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define a fuzzy set f in X as follows

$$f : X \to [0,1], \quad x \mapsto \begin{cases} t_0 & \text{if } x = 0, \\ t_1 & \text{if } x = c, \\ t_2 & \text{if } x \in \{a, b\} \end{cases}$$

where $t_0 < t_1 < t_2$ in $[0,1]$. It is routine to verify that f is an anti fuzzy commutative ideal of X.

THEOREM 3.18. *Every anti fuzzy commutative ideal is an anti fuzzy ideal.*

PROOF: Let f be an anti fuzzy commutative ideal of X. If we put $y = 0$ in (3.6), then

$$\begin{aligned} \max\{f(x*z), f(z)\} &= \max\{f((x*0)*z), f(z)\} \\ &\geq f(x*(0*(0*x))) = f(x). \end{aligned}$$

Hence f is an anti fuzzy ideal of X. □

The converse of Theorem 3.18 is not true as seen in the following example.

EXAMPLE 3.19. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Define a fuzzy set f in X as follows

$$f : X \to [0,1], \quad x \mapsto \begin{cases} s_0 & \text{if } x = 0, \\ s_1 & \text{if } x = 1, \\ s_2 & \text{if } x \in \{2,3,4\} \end{cases}$$

where $s_0 < s_1 < s_2$ in $[0,1]$. Then f is an anti fuzzy ideal of X. But it is not an anti fuzzy commutative ideal of X since

$$f(2 * (3 * (3 * 2))) \not\leq \max\{f(0), f((2 * 3) * 0)\}.$$

We provide a characterization of an anti fuzzy commutative ideal.

THEOREM 3.20. *For a fuzzy set f in X, the following are equivalent.*

(1) *f is an anti fuzzy commutative ideal of X.*
(2) *f is an anti fuzzy ideal of X satisfying the following condition*

$$(\forall x, y \in X)\,(f(x * (y * (y * x))) \leq f(x * y))\,. \tag{3.7}$$

PROOF: Assume that f is an anti fuzzy commutative ideal of X. Then f is an anti fuzzy ideal of X (see Theorem 3.18). Taking $z = 0$ in (3.6) and using (3.4) and (2.1), we have (3.7).

Conversely, suppose that (2) is valid. Then

$$f(x * y) \leq \max\{f((x * y) * z), f(z)\} \tag{3.8}$$

for all $x, y, z \in X$. Combining (3.7) and (3.8), we get (3.6). The proof is complete. □

DEFINITION 3.21 ([3]). *Let f be a fuzzy set in X. A number $t \in [0,1]$ is called a permeable I-value for f if $U(f;t) \neq \emptyset$ and the following assertion is valid.*

$$(\forall x, y \in X)\,(f(y) \geq t \;\Rightarrow \max\{f(y * x), f(x)\} \geq t)\,. \tag{3.9}$$

Definition 3.22. *Let f be a fuzzy set in X. A number $t \in [0,1]$ is called a permeable C-value for f if $U(f;t) \neq \emptyset$ and the following assertion is valid.*

$$f(x*(y*(y*x))) \geq t \Rightarrow \max\{f((x*y)*z), f(z)\} \geq t \tag{3.10}$$

for all $x, y, z \in X$.

Example 3.23. *Consider a BCK-algebra $X = \{0, a, b, c\}$ which is given in Example 3.17. Let f be a fuzzy set in X defined by $f(0) = 0.3$, $f(a) = f(b) = 0.7$ and $f(c) = 0.5$. If $t \in (0.5, 0.7]$, then $U(f;t) = \{a, b\}$ and it is easy to check that t is a permeable C-value for f.*

Theorem 3.24. *Let f be a fuzzy commutative ideal of X. If $t \in [0,1]$ is a permeable C-value for f, then the nonempty upper t-level set $U(f;t)$ is a C-energetic subset of X.*

Proof: Assume that $U(f;t) \neq \emptyset$ for $t \in [0,1]$. Let $x, y \in X$ be such that

$$x*(y*(y*x)) \in U(f;t).$$

Then $f(x*(y*(y*x))) \geq t$, and so $\max\{f((x*y)*z), f(z)\} \geq t$ by (3.10). It follows that $f((x*y)*z) \geq t$ or $f(z) \geq t$, that is, $(x*y)*z \in U(f;t)$ or $z \in U(f;t)$. Hence $\{z, (x*y)*z\} \cap U(f;t) \neq \emptyset$, and therefore $U(f;t)$ is a C-energetic subset of X. □

Since $U(f;t) \cup L^*(f;t) = X$ and $U(f;t) \cap L^*(f;t) = \emptyset$ for all $t \in [0,1]$, we have the following corollary.

Corollary 3.25. *Let f be a fuzzy commutative ideal of X. If $t \in [0,1]$ is a permeable C-value for f, then $L^*(f;t)$ is empty or a commutative ideal of X.*

Theorem 3.26. *For a fuzzy set f in X, if there exists a subset K of $[0,1]$ such that $\{U(f;t), L^*(f;t)\}$ is a partition of X and $L^*(f;t)$ is a commutative ideal of X for all $t \in K$, then t is a permeable C-value for f.*

Proof: Assume that $f(x*(y*(y*x))) \geq t$ for any $x, y \in X$. Then

$$x*(y*(y*x)) \in U(f;t),$$

and so $\{z, (x*y)*z\} \cap U(f;t) \neq \emptyset$ since $U(f;t)$ is a C-energetic subset of X. It follows that $z \in U(f;t)$ or $(x*y)*z \in U(f;t)$ and so that

$$\max\{f((x*y)*z), f(z)\} \geq t.$$

Therefore t is a permeable C-value for f. □

Theorem 3.27. *Let f be a fuzzy set in X with $U(f;t) \neq \emptyset$ for $t \in [0,1]$. If f is an anti fuzzy commutative ideal of X, then t is a permeable C-value for f.*

Proof: Let $x, y, z \in X$ be such that $f(x * (y * (y * x))) \geq t$. Then

$$t \leq f(x * (y * (y * x))) \leq \max\{f((x * y) * z), f(z)\}$$

by (3.6). Hence t is a permeable C-value for f. □

Theorem 3.28. *If f is an anti fuzzy commutative ideal of X, then*

$$(\forall t \in [0,1])\,(U(f;t) \neq \emptyset \;\Rightarrow\; U(f;t) \text{ is a } C\text{-energetic subset of } X)\,.$$

Proof: Let $x, y, z \in X$ be such that $x * (y * (y * x)) \in U(f;t)$. Then

$$f(x * (y * (y * x))) \geq t,$$

which implies from (3.6) that

$$t \leq f(x * (y * (y * x))) \leq \max\{f((x * y) * z), f(z)\}.$$

Hence $f((x*y)*z) \geq t$ or $f(z) \geq t$, that is, $(x*y)*z \in U(f;t)$ or $z \in U(f;t)$. Thus $\{z, (x * y) * z\} \cap U(f;t) \neq \emptyset$, and therefore $U(f;t)$ is a C-energetic subset of X. □

Theorem 3.29. *For any fuzzy set f in X, every permeable C-value for f is a permeable I-value for f.*

Proof: Let $t \in [0,1]$ be a permeable C-value for f. Assume that $f(y) \geq t$ for all $y \in X$. Then

$$t \leq f(y) = f((y * (0 * (0 * y)))$$

by (V) and (2.1), and so

$$t \leq \max\{f((y * 0) * z), f(z)\} = \max\{f(y * z), f(z)\}$$

for all $y, z \in X$ by (3.10) and (2.1). Therefore t is a permeable I-value for f. □

Definition 3.30 ([3]). *Let f be a fuzzy set in X. A number $t \in [0,1]$ is called an anti permeable I-value for f if $L(f;t) \neq \emptyset$ and the following assertion is valid.*

$$(\forall x, y \in X)\,(f(y) \leq t \;\Rightarrow \min\{f(y * x), f(x)\} \leq t)\,. \tag{3.11}$$

Theorem 3.31. *Let f be a fuzzy set in X with $L(f;t) \neq \emptyset$ for $t \in [0,1]$. If f is a fuzzy ideal of X, then t is an anti permeable I-value for f.*

Proof: Let $f(y) \leq t$ for $y \in X$. Then

$$\min\{f(y * x), f(x)\} \leq f(y) \leq t$$

for all $x \in X$ by (2.10). Hence t is an anti permeable I-value for f. □

DEFINITION 3.32. *Let f be a fuzzy set in X. A number $t \in [0,1]$ is called an anti permeable C-value for f if $L(f;t) \neq \emptyset$ and the following assertion is valid.*

$$f(x*(y*(y*x))) \leq t \;\Rightarrow \min\{f((x*y)*z), f(z)\} \leq t \tag{3.12}$$

for all $x, y, z \in X$.

THEOREM 3.33. *Let f be a fuzzy set in X with $L(f;t) \neq \emptyset$ for $t \in [0,1]$. If f is a fuzzy commutative ideal of X, then t is an anti permeable C-value for f.*

PROOF: Let $x, y \in X$ be such that $f(x*(y*(y*x))) \leq t$. Then

$$\min\{f((x*y)*z), f(z)\} \leq f(x*(y*(y*x))) \leq t$$

for all $z \in X$ by (2.12). Hence t is an anti permeable C-value for f. □

THEOREM 3.34. *Let f be an anti fuzzy commutative ideal of X. If $t \in [0,1]$ is an anti permeable C-value for f, then the lower t-level set $L(f;t)$ is a C-energetic subset of X.*

PROOF: Let $x, y \in X$ be such that $x*(y*(y*x)) \in L(f;t)$. Then $f(x*(y*(y*x))) \leq t$ and so $\min\{f((x*y)*z), f(z)\} \leq t$ by (3.12). It follows that $(x*y)*z \in L(f;t)$ or $z \in L(f;t)$. Hence $\{z, (x*y)*z\} \cap L(f;t) \neq \emptyset$, and therefore $L(f;t)$ is a C-energetic subset of X. □

COROLLARY 3.35. *Let f be an anti fuzzy commutative ideal of X. If $t \in [0,1]$ is an anti permeable C-value for f, then $U^*(f;t)$ is empty or a commutative ideal of X.*

THEOREM 3.36. *For a fuzzy set f in X, if there exists a subset K of $[0,1]$ such that $\{U^*(f;t), L(f;t)\}$ is a partition of X and $U^*(f;t)$ is a commutative ideal of X for all $t \in K$, then t is an anti permeable C-value for f.*

PROOF: Assume that $f(x*(y*(y*x))) \leq t$ for any $x, y \in X$. Then

$$x*(y*(y*x)) \in L(f;t),$$

and so $\{z, (x*y)*z\} \cap L(f;t) \neq \emptyset$ for all $z \in X$ since $L(f;t)$ is a C-energetic subset of X. It follows that $f(z) \leq t$ or $f((x*y)*z) \leq t$, and so that

$$\min\{f((x*y)*z), f(z)\} \leq t.$$

Therefore t is an anti permeable C-value for f. □

Acknowledgement

The authors wish to thank the anonymous reviewer(s) for the valuable suggestions.

References

[1] S. M. Hong and Y. B. Jun, *Anti fuzzy ideals in BCK-algebras*, **Kyungpook Math. J.** 38 (1998), pp. 145–150.
[2] Y. Huang, **BCI-algebra**, Science Press, Beijing, 2006.
[3] Y. B. Jun, S. S. Ahn and E. H. Roh, *Energetic subsets and permeable values with applications in BCK/BCI-algebras,* **Appl. Math. Sci.** 7 (2013), no. 89, pp. 4425–4438.
[4] Y. B. Jun and E. H. Roh, *Fuzzy commutative ideals of BCK-algebras*, **Fuzzy Sets and Systems** 64 (1994), pp. 401–405.
[5] J. Meng, *Commutative ideals in BCK-algebras*, **Pure Appl. Math.** (in China) 9 (1991), pp. 49–53.
[6] J. Meng and Y. B. Jun, **BCK-algebras**, Kyungmoonsa Co. Seoul, Korea (1994).
[7] L. A. Zadeh, *Fuzzy sets*, **Information and Control** 8 (1965), pp. 338–353.

Department of Mathematics Education (and RINS)
Gyeongsang National University, Jinju 52828, Korea
e-mail: skywine@gmail.com

Department of Mathematics Education,
Chinju National University of Education, Jinju 660-756, Korea
e-mail: idealmath@gmail.com

Department of Mathematics,
Jeju National University, Jeju 690-756, Korea
e-mail: szsong@jejunu.ac.kr