Bulletin of the **S**ection of **L**ogic
Volume 53/3 (2024), pp. 275–299
https://doi.org/10.18778/0138-0680.2024.05

Leonard Kupś
Szymon Chlebowski

MEANING IS USE: THE CASE OF PROPOSITIONAL IDENTITY

Abstract

We study natural deduction systems for a fragment of intuitionistic logic with propositional identity from the point of view of proof-theoretic semantics. We argue that the identity connective is a natural operator to be treated under the *elimination rules as basic* approach.

Keywords: intuitionistic Logic, non-Fregean logic, proof-theoretic semantics.

2020 Mathematical Subject Classification: 03B20, 03B60, 03F03.

1. Introduction

The main idea behind proof-theoretic semantics is to view the meaning of a logical connective as given by the conditions under which a corresponding proposition *can be asserted*[1]. This approach is related to the Wittgenstein's slogan that *meaning is use*, contrary to the traditional view that meaning is given by the *truth conditions*. From the point of view of natural deduction, there are two kinds of rules: introduction rules and elimination rules. Let us start with the famous observation by Gentzen:

[1] A thorough exposition of this approach is presented in the work of [12], [4] and [14] among many others.

Presented by: Nils Kürbis
Received: October 11, 2022
Published online: May 20, 2024

> ”The introductions represent, as it were, the ’definitions’ of the symbol concerned, and the eliminations are no more, in the final analysis, than the consequences of those definitions.” [8, p. 80]

Introduction rules specify the conditions under which a proposition of a certain form can be asserted, while elimination rules state the requirements pertaining to what can be deduced from a given proposition. Most intuitionistic connectives, such as implication, have both introduction and elimination rules. The *Falsum* constant is an exception here—it has only an elimination rule. The reason is that Falsum cannot be asserted under any conditions, but hypothetically we need to know what can be deduced from it (in our case, every proposition). Thus, the meaning of the Falsum constant is established only by the elimination rule. In this paper we will try to show that a propositional identity connective can be treated in a similar manner.

In the next section, fundamental notions will be introduced, concerning both the logic used throughout the paper and proof-theoretic semantics in general. Then we shall turn to the definition of validity based on introductory rules. In the following subsection, a number of examples will be examined. The third section concerns validity with elimination rules as basic, and it is structured analogically to the previous one: firstly we define such validity then we provide examples. The final section is a brief conclusion.

2. Intuitionistic Logic with Identity

In intuitionistic terms, we are not interested in propositions being true or false but in *constructions* which prove them. Equivalence of two formulae, A and B, means that every proof of A can be transformed into a proof of B and *vice versa*. Thus, whenever A is provable B is provable as well. However, it is interesting to consider a stronger notion which says that the classes of constructions proving A and B are *exactly* the same. This is the intended interpretation of the propositional identity connective on the grounds of intuitionistic logic.

2.1. BHK

What follows is a version of the BHK-interpretation of Falsum ($\bot$), intuitionistic implication ($\supset$) and propositional identity ($\equiv$).

there is no proof of $\bot$	
a is a proof of $A \supset B$	a is a construction that converts each proof a_1 of A into a proof $a_2(a_1)$ of B
a is a proof of $A \equiv B$	a is the identity function

Here a formula A may be thought of as representation of the set of its own proofs.

The additional condition for identity can be interpreted in the following ways:

- a proof of $A \equiv B$ is the identity function transforming a given proof of A into a proof of B;
- a proof of $A \equiv B$ establishes the fact that two sets of proofs are equal.

Both interpretations use the notion of identity function, but differently typed—one of them transforms proofs, the other one sets of proofs. We do not claim that every proof of A can be transformed into a proof of B by the identity function, since we want formula $A \equiv A$ to be valid under our interpretation of $\equiv$ (naturally there may be non-normal proofs of A which are not identical to a normal one).

Naturally, identity is stronger than implication: if we have an arbitrary proof of $A \equiv B$ it will also count as a proof of $A \supset B$ and $B \supset A$ (due to symmetry of $\equiv$).

2.2. Hilbert-style formalization

The logic we are going to consider can be thought of as an intuitionistic variant of basic non-fregean logic (SCI—*Sentential Calculus with Identity*) introduced by [1]. We call it ISCI—*Intuitionistic Sentential Calculus with Identity*.

The language $\mathcal{L}_{\mathsf{ISCI}}$ of the logic ISCI is defined by the following grammar:

$$A \ ::= \ V \mid \bot \mid \ A\&A \mid A \vee A \mid A \supset A \mid A \equiv A$$

where V is a denumerable set of propositional variables. The axiom system for ISCI can be obtained from any such system for INT by the addition of

$\equiv$-specific axioms (see Table 1). The first axiom underlines that identity is reflexive; the second axiom shows identity as a stronger connective than implication; and the third axiom expresses the fact that $\equiv$ is a congruence relation. The axioms are valid under the proposed interpretation of the identity connective, as it has been shown in [3]. The only rule of inference is *modus ponens.*

Table 1. Axioms for propositional identity; $\otimes \in \{\&, \vee, \supset, \equiv\}$

1. $A \equiv A$
2. $(A \equiv B) \supset (A \supset B)$
3. $(A \equiv B) \supset ((C \equiv D) \supset ((A \otimes C) \equiv (B \otimes D)))$

2.3. Natural deduction—synthetic approach

Following the prevailing *meaning is use* paradigm, throughout the paper we use the framework of natural deduction.

The first natural deduction system for ISCI we consider closely follows the corresponding axiom system. We use standard natural deduction rules for intuitionistic logic adding three specific rules from Table 2. The notation $[A \equiv A^n]^j$ indicates that the assumption $A \equiv A$ is discharged, n indicates the number of instances of a formula that are closed and j is the discharge label. This system was shown to be complete with respect to Hilbert-style system and enjoys normalisation [2].

According to the well-known Gentzen's idea the meaning of each connective is fixed by its introduction rule(s) and corresponding elimination rules are somehow justified by means of introduction rules. Here all rules are *general elimination rules* [11][2]. They are formulated in a general form: conclusions do not have specified logical form. Thus, there are possible applications of elimination rules for a given connective which introduce a formula with the same connective as a main sign.

[2] Althought rule $\equiv_1$ would be considered *general introduction rule* in [10], we prefer to consider it as a general elimination rule, since the formula introduced in the conclusion does not have specified logical form. But certainly there are *introductory applications* of this rule, that is, application which introduces identity.

Table 2. Identity specific rules in the system $\mathbf{ND}_{\mathsf{ISCI}}$ $(0 \leq n)$

$$\dfrac{\begin{array}{c}[A \equiv A^n]^j \\ \vdots \\ C\end{array}}{C}\ \equiv_1, j \qquad \dfrac{A \equiv B \qquad \begin{array}{c}[A \supset B^n]^j \\ \vdots \\ C\end{array}}{C}\ \equiv_2, j$$

$$\dfrac{A \equiv C \quad B \equiv D \qquad \begin{array}{c}[(A \otimes B) \equiv (C \otimes D)^n]^j \\ \vdots \\ F\end{array}}{F}\ \equiv_3, j$$

However, these rules reflect an important feature of propositional identity: it is intensional and it cannot be established solely on the fact that both of its components are provable. Thus, we cannot *synthesise* propositional identity and we do not know how to introduce it with one exception—one can safely assume reflexive identity, since assumption of this form can always be closed. On the other hand there is a specific rule for synthesizing more complex identities from simpler ones (due to the importance of this rule we call it *synthetic approach* to identity). However, it does not entail that this connective has no meaning: these rules give us hints on how to proceed when we have already established that some identity holds. Thus, we know how to use it and, according to Wittgenstein's slogan, it has meaning.

2.4. Natural deduction—analytic approach

The system $\mathsf{ND}_{\mathsf{ISCI}}$ introduced in the previous section is closely related to the axiomatic formulation of ISCI. Natural deduction rules in this system correspond to axioms. Yet, since we know that the symbol $\equiv$ is semantically interpreted as equality (in the classical version, the SCI system) or as identity function in ISCI, we can treat identity in a similar manner as equality is treated in First-Order Logic (FOL). The rules are presented in Table 3.

Table 3. Identity specific rules in the system $\mathbf{ND}^{*}_{\mathsf{ISCI}}$ $(0 \leq n)$

$$\frac{[A]^u}{A \equiv A} \equiv I, u \qquad \frac{A \equiv B \quad \phi(A) \quad \begin{matrix}[\phi(B/A)^n]^j \\ \vdots \\ F\end{matrix}}{F} \equiv E, j$$

On the left-hand side we have the introduction rule for identity: having established A we can conclude that $A \equiv A$ at the same time discharging the open assumption A[3]. According to the elimination rule, if we have established that $A \equiv B$ and we have a formula ϕ with at least one occurrence of the formula A (we indicate that there exists such occurrence by $\phi(A)$), we can conclude formula $\phi(B/A)$, that is the formula ϕ with at least one occurrence of A replaced by B. Due to the central character of the elimination rule, which enables replacing identical subformulas in a given formula, we call this approach to identity *analytic*.

Contrary to the synthetic approach to ISCI, the present set of rules is compatible with Gentzen's analysis of logical connectives: each connective has both introduction and elimination rule. Thus, a new detour is possible: the elimination rule for the identity connective has been applied just after the introduction rule for that connective:

$$\frac{\dfrac{[A]}{A \equiv A} \equiv I \qquad \begin{matrix}\vdots \\ \phi(A)\end{matrix}}{\phi(A/A)} \equiv E$$

[3] We choose this form of the introduction rule for $\equiv$ to exhibit the similarity between BHK-interpretations of implication and identity: both the former and the latter denote a function, but in case of identity it is a very specific one. Other possibility, since the assumption is immediately discharged, is to consider a no-premiss rule:

$$\frac{}{A \equiv A} \equiv I$$

This derivation can easily be transformed in such a way that an occurrence of $A \equiv A$ disappears from the derivation:

$$\begin{array}{c} \vdots \\ \phi(A) \end{array}$$

The system is complete and enjoys normalisation, see [2].

3. Validity based on introduction rules

Let us now recall some basic concepts of proof-theoretic semantics to serve as a rudiment of our further inquiry. First and foremost, it is convenient to think of proof-theoretic semantics in contrast to standard *model-theoretic semantics*. In standard semantics we start with some names and sentences which are represented by terms and formulae. Then we assign meaning to these objects and we specify truth conditions. Having done that we are finally able to define the notions of *validity* and *entailment*. In proof-theoretic semantics the starting point is the notion of an argument which can be represented as a formal object, most often as a *derivation* in a natural deduction system. The next step is to define the notion of validity of derivations and arguments which they represent. So, contrary to model-theoretic semantics, we build up semantic notions from an inferential point of view. Note that the validity of concrete natural deduction rules is established in terms of validity of derivations:

> (...) rules or consequences are regarded as steps which preserve the validity of arguments (...) [14, p. 529]

We shall start with some terminological remarks. By *derivation structure* (*proof skeleton* in Prawitz terms) we mean a logical representation of a certain type of arguments. It can be depicted as a natural deduction derivation, with a conclusion as root and formulae called assumptions as leaves, built from arbitrary rules of the form:

$$\dfrac{\begin{array}{c}[\Gamma_1]^i \\ \vdots \\ A_1\end{array} \quad \ldots \quad \begin{array}{c}[\Gamma_n]^i \\ \vdots \\ A_n\end{array}}{B}\; R, i$$

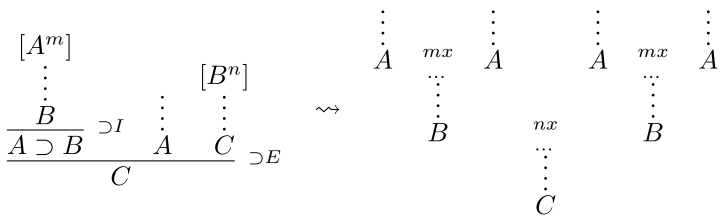
Note that derivation structures may not be properly built derivations in one of the natural deduction systems we have just defined. If all assumptions within a given derivation structure are discharged then it is *closed* [14, p. 530]. Otherwise, a derivation structure is *open*. A *canonical* derivation structure ends with an application of an introduction rule [6, p. 36].

The notion of validity is relativised to an *atomic system* S and a *reduction system* $\mathcal{J}$. By the *atomic system* S we understand a logic-free system with production rules for atomic formulae [14, p. 542], which correspond to production systems of grammars [16]. In our case the atomic formulae are propositional variables and identities. By the *reduction system* we mean a system of meta-rules enabling transformation of one derivation structure into another. Look at normalisation of derivations as an example of a reduction system. The *detour convertibility* serves to exclude, in the given derivation, consecutive pairs of introduction and elimination rules applications for the same connective. The *permutation convertibility* allows a rearrangement of assumptions if an instance of an elimination rule has a major premiss that is a conclusion of another elimination rule application. Examples of both normalisations are shown in Examples 2 and 3.

$$\frac{B \equiv A}{A \equiv B}\,{\scriptstyle sym} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\vdots \\ B \equiv A\end{matrix} \quad \dfrac{[B]^1}{B \equiv B}\,{\scriptstyle \equiv I.1} \quad [A \equiv B]^2}{A \equiv B}\,{\scriptstyle \equiv E.2}$$

Example 1. Rule *sym* cannot be justified using I-validity

There are two main approaches to the definition of validity of derivations (and some combinations of them; for an in-depth classification see [7]). One of them, which we will address first, closely follows Gentzen and assumes that introduction rules are meaning-giving and elimination rules need to be somehow justified based on introduction rules. Another approach, which we believe is more appropriate for our treatment of propositional identity, is based on the primacy of elimination rules. According to Schroeder-Heister the distinction between these two paradigms reflects the duality between *verificationism* and *falsificationism* [15].



Example 2. Detour convertibility example

The definition of the validity of a derivation is given below. Further, we are going to introduce yet another notion of validity and thenceforth, we are going to refer to this type of validity as *I-validity*.

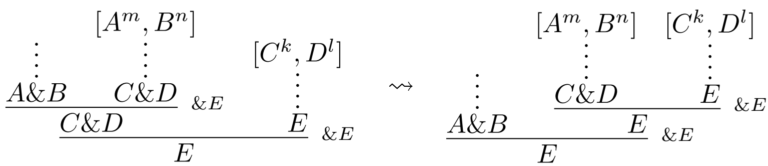
1. Every closed proof in the underlying atomic system is valid.

2. A closed canonical proof is considered valid, if its immediate subproofs are valid.

3. A closed non-canonical proof is considered valid, if it reduces to a valid closed canonical proof or to a closed proof in the atomic system.

4. An open proof is considered valid, if every closed proof obtained by replacing its open assumptions with closed proofs and its open variables with closed terms is valid [16].

The exact definition of validity based on introduction rules is formulated below. The S in the following definition refers to an arbitrary atomic system, $\mathcal{J}$ is a *justification*, that is a reduction system. Validity depends on the underlying atomic system S and on the type of reduction procedures used as well. S' is an *extension* of the system S if S' is S or S' results from adding further production rules to S.



Example 3. Permutation convertibility example

Definition 3.1 (I-validity).

1. Every closed derivation structure in S is S-valid with respect to $\mathcal{J}$ (for every $\mathcal{J}$).
2. A closed canonical derivation structure is S-valid with respect to $\mathcal{J}$, if its immediate substructure $\begin{array}{c} A \\ \vdots \\ B \end{array}$ is S-valid with respect to $\mathcal{J}$.
3. A closed non-canonical derivation structure is S-valid with respect to $\mathcal{J}$, if it reduces, with respect to $\mathcal{J}$, to a canonical derivation structure, which is S-valid, with respect to $\mathcal{J}$.
4. An open derivation structure

$$\begin{array}{c} A_1, \ldots, A_n \\ \vdots \\ B \end{array}$$

where all open assumptions are among $A_1, \ldots, A_n$ is S-valid with respect to $\mathcal{J}$, if for every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$, and for every list of i closed derivation structures $\begin{array}{c} \vdots \\ A_i \end{array}$ $(1 \leq i \leq n)$, which are S'-valid with respect to $\mathcal{J}'$,

$$\begin{array}{ccc} \vdots & & \vdots \\ A_1, & \ldots, & A_n \\ & \vdots & \\ & B & \end{array}$$

is S'-valid with respect to $\mathcal{J}'$ [15, pp. 162-163].

$$\frac{A \equiv B}{(A \supset B)\&(B \supset A)}\, R_1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{A\equiv B \quad [A]^2 \quad [B]^1}{B}\,{\scriptstyle \equiv E.1}}{A\supset B}\,{\scriptstyle \supset I.2} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\equiv B\end{array} \quad \dfrac{[A]^3}{A\equiv A}\,{\scriptstyle \equiv I.3} \quad [B\equiv A]^4}{B\equiv A}\,{\scriptstyle \equiv E.4} \quad [B]^6 \quad [A]^5}{A}\,{\scriptstyle \equiv E.5}}{B\supset A}\,{\scriptstyle \supset I.6}}{(A\supset B)\&(B\supset A)}\,{\scriptstyle \& I}$$

Example 4. Justification of R_1 using I-validity

Validity is first and foremost a feature of a derivation structure, but when we say that a natural deduction rule is valid, what we mean is that the corresponding one-step derivation structure is valid.

Examples of I-valid derivations In this section, we are going to examine validity as defined in the definition 3.1. Each example starts with a rule which is reduced to a valid derivation structure in the arbitrary underlying atomic system S. Every exemplary rule that we are going to address in this paper is an instance of an open derivation structure with exactly one open assumption. According to point (4) of definition 3.1 we are going to treat those open assumptions as follows: we extend S in such a way that the assumption in question can be derived from (at least one) valid derivation, and we proceed examining the given example as an instance of closed derivation structure.

We are going to focus on examples based on the analytic approach to ISCI that includes an introduction rule for the $\equiv$ connective. The example 1 shows a reduction with no open assumptions (excluding the initial one), yet impossible to be reduced to a canonical form—i.e. to introduce the $\equiv$ connective in the last step of the derivation. Thus, this derivation does not meet the point (iii) of definition 3.1 and therefore is not I-valid.

However, in the case of the derivation structure that includes the $\equiv$ connective but not as the main connective in the conclusion, as rule R_1 (see Ex. 4, p. 285), it is possible to reduce the derivation to the canonical form. Therefore, rule R_1 is I-valid, even though it contains *sym* derivation structure.

As we have seen the I-validity fails to recognise valid derivations that include the $\equiv$ connective as a primary connective in the conclusion. Thus, an approach based on introduction rules is unsatisfactory in the case of ISCI. Therefore, we are going to turn to the elimination rules based alternative in the next section.

4. Validity based on elimination rules

There exists a notion of e-canonicity of derivations [5, 7] that is closely related to the I-validity (3.1), which is unsatisfactory in our case for the same reasons that we have described in the previous section. In what follows, we are going to define e-validity in a different manner.

In the introductory section, we quoted Gentzen (p. 1), who perceived elimination rules as consequences of definitions given by the introduction rules for the given connective. Therefore, validity with elimination rules as basic views a derivation as valid if all immediate logical consequences, that can be derived from that derivation, are valid as well. As I-validity examines whether all the steps taken in the derivation to this point are legitimate, acting retrospectively in a sense, the E-validity is prospective, investigating the legitimacy of the steps that can be taken from the conclusion of the derivation: if all applications of elimination rules to the end formula of some derivation structure $\mathcal{D}$ result in E-valid derivation structures, then the initial derivation structure $\mathcal{D}$ is considered to be E-valid.

Validity based on elimination rules (*E-validity*) is defined as follows (adapted from [15, pp. 164–166]) for the synthetic approach to ISCI:

Definition 4.1 (E-validity).

1. Every closed derivation in S is *E-valid* with respect to $\mathcal{J}$, (for every $\mathcal{J}$).

2. (&) A closed derivation structure $\begin{array}{c}\vdots\\ A\&B\end{array}$ is E-valid in S with respect to $\mathcal{J}$, if the closed derivation structure

$$\dfrac{\begin{array}{c}\vdots\\ A\&B\end{array} \qquad \begin{array}{c}[A,B]^1\\ \vdots\\ C\end{array}}{C}\ \&E.1$$

is E-valid in S with respect to $\mathcal{J}$, or reduces to derivation structures which are E-valid in S with respect to $\mathcal{J}$.

($\supset$) A closed derivation structure $\begin{matrix}\vdots\\ A\supset B\end{matrix}$ is E-valid in S with respect to $\mathcal{J}$, if for every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$ and for every closed derivation structure $\begin{matrix}\vdots\\ A\end{matrix}$ which is E-valid in S with respect to $\mathcal{J}'$, the (closed) derivation structure

$$\dfrac{\begin{matrix}\vdots\\ A\supset B\end{matrix}\quad \begin{matrix}\vdots\\ A\end{matrix}\quad \begin{matrix}[A]^1\\ \vdots\\ C\end{matrix}}{C}\ \supset E.1$$

is E-valid in S' with respect to $\mathcal{J}'$, or reduces to derivation structures which are E-valid in S' with respect tp $\mathcal{J}'$.

($\vee$) A closed derivation structure $\begin{matrix}\vdots\\ A\vee B\end{matrix}$ is E-valid in S with respect to $\mathcal{J}$, if for every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$, and for all derivation structures $\begin{matrix}A\\ \vdots\\ C\end{matrix}$ and $\begin{matrix}B\\ \vdots\\ C\end{matrix}$ with atomic C, which are E-valid in S' with respect to $\mathcal{J}'$ and which depend on no assumptions beyond A and B, respectively, the (closed) derivation structure

$$\dfrac{A\vee B\quad \begin{matrix}[A]^1\\ \vdots\\ C\end{matrix}\quad \begin{matrix}[B]^1\\ \vdots\\ C\end{matrix}}{C}\ \vee E.1$$

is E-valid in S' with respect to $\mathcal{J}'$, or reduces to derivation structure, which is E-valid in S' with respect tp $\mathcal{J}'$.

(≡) i. A closed derivation structure $\begin{array}{c}\vdots\\ A\equiv B\end{array}$ is E-valid in S with respect to $\mathcal{J}$, if for every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$, and for every closed derivation structure $\begin{array}{c}A\supset B\\ \vdots\\ C\end{array}$ with atomic C, which is E-valid in S' with respect to $\mathcal{J}'$ and which depends on no assumptions beyond $A \supset B$, the (closed) derivation structure

$$\dfrac{\begin{array}{c}\vdots\\ A\equiv B\end{array}\qquad \begin{array}{c}[A\supset B]^1\\ \vdots\\ C\end{array}}{C}\ \equiv_2 .1$$

is E-valid in S' with respect to $\mathcal{J}'$, or reduces to derivation structure, which is E-valid in S' with respect to $\mathcal{J}'$.

ii. A closed derivation structure $\begin{array}{c}\vdots\\ A\equiv C\end{array}\ \begin{array}{c}\vdots\\ B\equiv D\end{array}$ is E-valid in S with respect to $\mathcal{J}$, if for every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$, and for every closed derivation structure $\begin{array}{c}(A\otimes B)\equiv(C\otimes D)\\ \vdots\\ F\end{array}$ with atomic F, which is E-valid in S' with respect to $\mathcal{J}'$ and which depends on no assumptions beyond $(A\otimes B) \equiv (C\otimes D)$, the (closed) derivation structure

$$\dfrac{\begin{array}{c}\vdots\\ A\equiv C\end{array}\quad \begin{array}{c}\vdots\\ B\equiv D\end{array}\qquad \begin{array}{c}[(A\otimes B)\equiv(C\otimes D)]^1\\ \vdots\\ F\end{array}}{F}\ \equiv_3 .1$$

is E-valid in S' with respect to $\mathcal{J}'$, or reduces to derivation structure, which is E-valid in S' with respect to $\mathcal{J}'$.

3. A closed derivation structure $\begin{array}{c}\vdots\\ A\end{array}$ of an atomic formula A, which is not a derivation in S, is E-valid in S with respect to $\mathcal{J}$, if it reduces with respect to $\mathcal{J}$ to a derivation in S.

4. An open derivation structure

$$\begin{array}{c} A_1, \ldots, A_n \\ \vdots \\ B \end{array}$$

where all open assumptions are among $A_1, \ldots, A_n$ is E-valid in S with respect to $\mathcal{J}$, if for every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$, and for every list of i closed derivation structures $\begin{array}{c}\vdots \\ A_i\end{array}$ $(1 \leq i \leq n)$, which are valid in S' with respect to $\mathcal{J}'$,

$$\begin{array}{c} \vdots \qquad\qquad \vdots \\ A_1, \quad \ldots, \quad A_n \\ \vdots \\ B \end{array}$$

is valid in S' with respect to $\mathcal{J}'$.

The definition based on the analytic approach to ISCI is the same as def. 4.1 with different rule for $\equiv$:

DEFINITION 4.2. ($\equiv^*$) A closed derivation structure $\begin{array}{c}\vdots \\ A \equiv B\end{array}$ is E-valid in S with respect to $\mathcal{J}$, if for every extension S' of S and every extension $\mathcal{J}'$ of $\mathcal{J}$, and for all closed derivation structures $\begin{array}{c}\vdots \\ \phi(A)\end{array}$ which are E-valid in S with respect to $\mathcal{J}$, the (closed) derivation structure

$$\dfrac{\begin{array}{c}\vdots \\ A \equiv B\end{array} \quad \begin{array}{c}\vdots \\ \phi(A)\end{array} \quad \begin{array}{c}[\phi(B/A)] \\ \vdots \\ F\end{array}}{F} \equiv E$$

is E-valid in S' with respect to $\mathcal{J}'$, or reduces to derivation structures, which are E-valid in S' with respect to $\mathcal{J}'$.

$$\dfrac{B \equiv A}{A \equiv B}\,sym \quad \rightsquigarrow \quad \dfrac{\dfrac{B \equiv A}{A \equiv B}\,sym \qquad \begin{array}{c}\vdots\\ \phi(A)\end{array} \qquad \begin{array}{c}[\phi(B/A)]^1\\ \vdots\\ C\end{array}}{C}\,{\equiv}E.1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ B \equiv A\end{array} \quad \dfrac{[B]^1}{B \equiv B}\,{\equiv}I.1 \quad [A \equiv B]^2}{A \equiv B}\,{\equiv}E.2 \qquad \begin{array}{c}\vdots\\ \phi(A)\end{array} \qquad \begin{array}{c}[\phi(B/A)]^3\\ \vdots\\ C\end{array}}{C}\,{\equiv}E.3$$

Example 5. Justification of *sym* using E-validity and analytic approach to identity.

Examples of E-valid derivations In this section, we will analyse E-validity of some natural deductions rules by providing appropriate justifications. Just as in the case of examples for I-validity, rules are instances of open derivations with exactly one open assumption each. According to point (4) of definition 4.1 we are going to proceed with reductions of those derivations assuming that there exists (at least) one valid derivation for the open assumption in question, analogically to the I-validity examples.

When examining the validity concerning definition 4.1 we start with an assumption that the rule in question is valid. Then from the conclusion of that rule an atomic formula is derived, with an application of an elimination rule for the main connective in the conclusion. In the next step, the formula C is derived from the premiss of the given rule. If the last reduction is successful the rule is valid according to E-validity.

The example of rule *sym* in the analytic approach to the ISCI was not I-valid (see Ex. 1). However, it proves to be E-valid (see Ex. 5). In the first step we assume that the rule *sym* is valid and we apply the elimination rule for the $\equiv$ connective: therefore, we assume that (1) there is a valid closed derivation from which we can conclude a formula ϕ with at least one occurrence of formula A, and (2) there is a valid close derivation with formula $\phi(B)$ (that is a formula ϕ with at least one occurrence of A replaced by B) as a (discharged) assumption and C as a conclusion; and we derive a conclusion C. In the next step, analogically to previous examples we derive C solely from the premiss of the rule *sym*. There are no additional open assumptions, therefore rule *sym* is valid.

An analogical method is applied in the case of E-valid rule and R_2 (see Ex. 6).

$$\dfrac{A \equiv B}{(A\&B) \equiv (B\&A)}\,R_2 \quad \rightsquigarrow \quad \dfrac{\dfrac{A \equiv B}{(A\&B) \equiv (B\&A)}\,R_2 \qquad \begin{array}{c}\vdots \\ \phi(A\&B)\end{array} \qquad \begin{array}{c}[\phi((B\&A)/(A\&B))]^1 \\ \vdots \\ C\end{array}}{C}\,\equiv E.1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ A \equiv B\end{array} \quad \dfrac{[A\&A]^1}{(A\&A) \equiv (A\&A)}\,(\equiv I.1) \quad [(A\&B) \equiv (B\&A)]^2}{(A\&B) \equiv (B\&A)}\,\equiv E.2 \qquad \begin{array}{c}\vdots \\ \phi(A\&B)\end{array} \qquad \begin{array}{c}[\phi((B\&A)/(A\&B))]^3 \\ \vdots \\ C\end{array}}{C}\,\equiv E.3$$

Example 6. Justification of R_2 using E-validity and analytic approach to identity.

In the case of invalid rule R_3 (see Ex. 7), to conclude C the formula $A \equiv B$ is needed but we cannot discharge it as an assumption. Thus, the derivation structure is no longer closed and the rule R_3 is E-invalid.

$$\dfrac{A\&B}{(A\&B) \equiv (B\&A)}\,\cancel{R_3} \quad \rightsquigarrow \quad \dfrac{\dfrac{A\&B}{(A\&B) \equiv (B\&A)}\,\cancel{R_3} \qquad \begin{array}{c}\vdots \\ \phi(A\&B)\end{array} \qquad \begin{array}{c}[\phi((B\&A)/(A\&B))]^1 \\ \vdots \\ C\end{array}}{C}\,\equiv E.1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\cancel{\begin{array}{c}\vdots \\ A \equiv B\end{array}} \quad \dfrac{[A\&A]^1}{(A\&A) \equiv (A\&A)}\,(\equiv I.1) \quad [(A\&B) \equiv (B\&A)]^2}{(A\&B) \equiv (B\&A)}\,\equiv E.2 \qquad \begin{array}{c}\vdots \\ \phi(A\&B)\end{array} \qquad \begin{array}{c}[\phi((B\&A)/(A\&B))]^3 \\ \vdots \\ C\end{array}}{C}\,\equiv E.3$$

Example 7. R_3 cannot be justified using E-validity and analytic approach to identity.

We can also analyse these rules in the synthetic approach. The steps taken in the reductions are very similar to the analytic approach. In the case of the rule *sym* (see Ex. 8), we begin by assuming that the rule in question is valid and apply the $\equiv .2$ elimination rule to the conclusion: thus, we assume that there is at least one closed, valid derivation from which we

$$\frac{B \equiv A}{A \equiv B}\, sym \quad \rightsquigarrow \quad \frac{\dfrac{B \equiv A}{A \equiv B}\, sym \qquad \begin{array}{c}[A \supset B]^1 \\ \vdots \\ C\end{array}}{C}\, {\equiv}2.1 \quad \rightsquigarrow$$

$$\frac{\dfrac{\dfrac{B \equiv A \qquad \dfrac{[B \equiv B]^1}{B \equiv B}\,{\equiv}1.1 \qquad [(B \equiv B) \equiv (A \equiv B)]^2}{(B \equiv B) \equiv (A \equiv B)}\,{\equiv}3.2 \qquad [(B \equiv B) \supset (A \equiv B)]^3}{(B \equiv B) \supset (A \equiv B)}\,{\equiv}2.3 \qquad \dfrac{[B \equiv B]^4}{B \equiv B}\,{\equiv}1.4 \qquad \dfrac{[A \equiv B]^6 \qquad \begin{array}{c}[A \supset B]^5 \\ \vdots \\ C\end{array}}{C}\,{\equiv}2.5}{C}\,{\supset}.6$$

Example 8. Justification of *sym* using E-validity and synthetic approach to identity.

can conclude atomic C from (closed) assumption $A \supset B$, and we conclude C. Then, we attempt to conclude C from the premiss of the *sym* rule. Even though the derivation is rather complex we are successful—there are no open assumptions, excluding the initial one—the rule *sym* is E-valid.

Analogically, we can prove that the rule R_2 is E-valid (Ex. 9 p. 293).

Interestingly, the rule R_3 in the synthetic approach (Ex. 10) fails to meet the criteria of E-validity for the same reasons as in the analytic approach. In the last step of the reduction, one of the assumptions is open. Therefore, the rule R_3 cannot be reduced to a closed derivation structure of required form and is not E-valid.

5. Object identity and propositional identity

Since we are interested in proof-theoretical treatment of *propositional identity* connective it would be helpful to look into, at first sight analogous, characterization of *object identity*. There is an ongoing debate about the proof-theoretical characterization of it. Usually, one can add two rules for a given natural deduction system for First-Order Logic (see [11]).

$$\frac{}{a = a}\, ref$$

$$\frac{a = b \quad Pa}{Pb}\, rep$$

$$\frac{A \equiv B}{(A\&B) \equiv (B\&A)}\, R_2 \quad \rightsquigarrow \quad \frac{\dfrac{A \equiv B}{(A\&B) \equiv (B\&A)}\, R_2 \qquad \begin{matrix}[(A\&B) \supset (B\&A)]^1 \\ \vdots \\ C\end{matrix}}{C}\, {\equiv}2.1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \equiv B \qquad \dfrac{[A \equiv A]^1}{A \equiv A}\,{\scriptstyle \equiv 1.1} \qquad [(A \equiv A) \equiv (B \equiv A)]^2}{(A \equiv A) \equiv (B \equiv A)}\,{\scriptstyle \equiv 3.2} \qquad [(A \equiv A) \supset (B \equiv A)]^3}{(A \equiv A) \supset (B \equiv A)}\,{\scriptstyle \equiv 2.3} \qquad \dfrac{[A \equiv A]^4}{A \equiv A}\,{\scriptstyle \equiv 1.4} \qquad [B \equiv A]^5}{B \equiv A}\,{\scriptstyle \supset .5} \qquad A \equiv B \qquad [(A\&B) \equiv (B\&A)]^6}{(A\&B) \equiv (B\&A)}\,{\scriptstyle \equiv 3.6} \qquad [(A\&B) \supset (B\&A)]^7}{C}\,{\scriptstyle \equiv 2.7}$$

Example 9. Justification of R_2 using E-validity e-validity and synthetic approach to identity.

$$\dfrac{A\&B}{(A\&B)\equiv(B\&A)}\,\not{R_3} \quad\rightsquigarrow\quad \dfrac{\dfrac{A\&B}{(A\&B)\equiv(B\&A)}\,\not{R_3} \qquad \begin{matrix}[(A\&B)\supset(B\&A)]^1\\ \vdots \\ C\end{matrix}}{C}\,{\equiv}2.1 \quad\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A\equiv A]^1}{A\equiv A}\,{\equiv}1.1 \quad \dfrac{[B\equiv B]^2}{B\equiv B}\,{\equiv}1.2 \quad [(A\equiv B)\equiv(A\equiv B)]^3}{(A\equiv B)\equiv(A\equiv B)}\,{\equiv}3.3 \quad [(A\equiv B)\supset(A\equiv B)]^4}{(A\equiv B)\supset(A\equiv B)}\,{\equiv}2.4 \quad \cancel{\begin{matrix}\vdots\\ A\equiv B\end{matrix}} \quad \dfrac{[(A\equiv B)]^5}{(A\&B)\equiv(B\&A)}}{(A\&B)\equiv(B\&A)}\,{\supset}.5 \quad \dfrac{[(A\&B)\supset(B\&A)]^6}{C}}{C}\,{\supset}.6$$

Example 10. R_3 cannot be justified using E-validity and synthetic approach to identity.

The argument against such a treatment of identity is that it is against Gentzen's dictum that introduction rules for a given operator justify elimination rule(s) for it. It seems that ref does not justify rep, at least in a way analogous to the way the rules for the connectives like conjunction or implication do—introduction rule produces only reflexive identities, but elimination rule assume an arbitrary identity.

One of the rival propositions is to go back to Leibnizian laws of identity:

(P1) $\forall P \forall x, y((Px \supset\subset Py) \supset x = y)$—*identity of indiscernibles*

(P2) $x = y \supset (Px \supset\subset Py)$—*indiscernibility of identicals*

Intuitively, P1 gives us grounds for asserting identities while P2 enables us to infer something from it, when it has been already established [13]. The problem is that P1 requires Second-Order Logic to bind predicate variables. But we can somehow encode it in a natural deduction rule by means of a restriction of its use. The following rule:

$$\frac{\begin{array}{c}[Pa]\\ \vdots \\ Pb\end{array}}{a=b}\ \mathsf{P1}$$

can be used, provided P does not occur free in any assumption other than Pa. Then elimination rule follows from the introduction rule:

$$\frac{a=b \quad Pa}{Pb}\ \mathsf{P2}$$

and standard detour conversions can be applied—the following derivation

$$\frac{\dfrac{\begin{array}{c}[Pa]\\ \vdots \\ Pb\end{array}}{a=b}\ \mathsf{P1} \qquad Pa}{Pb}\ \mathsf{P2}$$

reduces to

$$\begin{array}{c}Pa\\ \vdots \\ Pb\end{array}$$

The thoughtful discussion of these two approaches can be found in [9].

Unfortunately, Leibnizian approach seems not to work in the context of propositional identity. Assume we were to accept the following rule (where $C(A)$ denotes a formula C having a formula A as a subformula):

$$\dfrac{\begin{matrix}[C(A)]\\ \vdots \\ C(B)\end{matrix}}{A \equiv B}\; I$$

with strong side condition that the rule can be applied if formula C (and any of its subformulas) does not occur in any assumption other than the one specified in the rule scheme. Then we would be able to prove $A\&B \equiv B\&A$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(A\&B)\&C]}{A\&B}}{B} \qquad \dfrac{\dfrac{[(A\&B)\&C]}{A\&B}}{A}}{B\&A} \qquad \dfrac{[(A\&B)\&C]}{C}}{(B\&A)\&C}}{A\&B \equiv B\&A}\; I$$

which should not be provable in the basic logic ISCI (it is considered valid in some of its extensions though).

6. Conclusions

In the context of pure intuitionistic logic proof-theoretic semantics based on elimination rules can be equivalent to semantics based on introduction rules. The differences arising from these two approaches are mostly of philosophical and procedural nature. However, it is more complicated in the context of ISCI. Propositional identity is different than other intuitionistic connectives. We can introduce intuitionistic disjunction having proved one of its disjuncts. Similarly, one can introduce intuitionistic implication when a derivation of the consequent from the antecedent is given. Yet, no formula of the form $A \equiv B$ can be obtained from its subformulae only, with the exception of $A \equiv A$. As we have seen on examples in this paper, it follows that the approach based on elimination rules works well in the extension of intuitionistic logic we have considered. It is also philosophically plausible—the fact that identity cannot be synthesised from its subformulae does not mean that we cannot hypothetically reason about identities and establish some of the desired properties, such as symmetry or transitivity. Moreover, the approach based on elimination rules is naturally compatible with Wittgenstein's dictum. When we are inside a certain *Sprachspiel* sometimes only decomposition rules (elimination rules) for certain operators exists. Consider equality between real numbers. There is no effective way of establishing that two reals are equal, but we can still claim that equality between real numbers is an equivalence relation. It would be interesting to compare these two paradigms using some other intensional propositional operators.

Acknowledgements. The work of the second author has been founded by National Science Centre, Poland, grant no. 2017/26/E/HS1/00127.

References

[1] S. L. Bloom, R. Suszko, *Investigations into the Sentential Calculus with Identity*, **Notre Dame Journal of Formal Logic**, vol. 13(3) (1972), pp. 289–308, DOI: https://doi.org/10.1305/ndjfl/1093890617.

[2] S. Chlebowski, M. Gawek, A. Tomczyk, *Natural Deduction Systems for Intuitionistic Logic with Identity*, **Studia Logica**, vol. 110(6) (2022), pp. 1381–1415, DOI: https://doi.org/10.1007/s11225-022-09995-0.

[3] S. Chlebowski, D. Leszczyńska-Jasion, *An investigation into intuitionistic logic with identity*, **Bulletin of the Section of Logic**, vol. 48(4) (2019), pp. 259–283, DOI: https://doi.org/10.18778/0138-0680.48.4.02.

[4] M. Dummett, **The logical basis of metaphysics**, Harvard University Press, Cambridge, Mass. (1991).

[5] N. Francez, *On the notion of canonical derivations from open assumptions and its role in proof-theoretic semantics*, **Review of Symbolic Logic**, vol. 8 (2015), pp. 296–305, DOI: https://doi.org/10.1017/S1755020315000027.

[6] N. Francez, **Proof-theoretic Semantics**, vol. 57 of Studies in Logic, College Publications, Rickmansworth (2015).

[7] N. Francez, *Views of proof-theoretic semantics: Reified proof-theoretic meanings*, **Journal of Logic and Computation**, vol. 26 (2016), pp. 479–494, DOI: https://doi.org/10.1093/logcom/exu035.

[8] G. Gentzen, *Investigations into logical deductions*, **American Philosophical Quarterly**, vol. 1(4) (1964), pp. 288–306, URL: http://www.jstor.org/stable/20009142.

[9] A. Klev, *The harmony of identity*, **Journal of Philosophical Logic**, vol. 48(5) (2019), pp. 867–884, DOI: https://doi.org/10.1007/s10992-018-09499-0.

[10] P. Milne, *Inversion principles and introduction rules*, [in:] H. Wansing (ed.), **Dag Prawitz on Proofs and Meaning**, Springer, Cham (2013), pp. 189–224, DOI: https://doi.org/10.1007/978-3-319-11041-7_8.

[11] S. Negri, J. von Plato, **Proof Analysis. A Contribution to Hilbert's Last Problem**, Cambridge University Press, Cambridge (2011), DOI: https://doi.org/10.1017/CBO9781139003513.

[12] D. Prawitz, *On the idea of a general proof theory*, **Synthese**, (1974), pp. 63–77, DOI: https://doi.org/10.1007/BF00660889.

[13] S. Read, *Identity and harmony*, **Analysis**, vol. 64(2) (2004), pp. 113–119, DOI: https://doi.org/10.1093/analys/64.2.113.

[14] P. Schroeder-Heister, *Validity Concepts in Proof-theoretic Semantics*, **Synthese**, vol. 148 (2006), pp. 525–571, DOI: https://doi.org/10.1007/s11229-004-6296-1.

[15] P. Schroeder-Heister, *Proof Theoretical Validity Based on Elimination Rules*, [in:] E. H. Haeusler, W. de Campos Sanz, B. Lopez (eds.), **Why is this a Proof? Festschrift for Luiz Carlos Pereira**, College Publications, Rickmansworth (2015), pp. 159–176.

[16] P. Schroeder-Heister, *Proof-Theoretic Semantics*, [in:] E. N. Zalta, U. Nodelman (eds.), **The Stanford Encyclopedia of Philosophy**, fall 2023 ed., Metaphysics Research Lab, Stanford University (2023), URL: https://plato.stanford.edu/archives/spr2018/entries/proof-theoretic-semantics/.

Leonard Kupś
Adam Mickiewicz University in Poznań, Poland
Department Psychology and Cognitive Science
ul. Szamarzewskiego 89/AB
60-568 Poznań, Poland
e-mail: leonard.kups@gmail.com

Szymon Chlebowski
Adam Mickiewicz University in Poznań, Poland
Department Psychology and Cognitive Science
ul. Szamarzewskiego 89/AB
60-568 Poznań, Poland
e-mail: szymon.chlebowski@amu.edu.pl