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A CATEGORY OF ORDERED ALGEBRAS EQUIVALENT TO THE CATEGORY OF MULTIALGEBRAS

Abstract

It is well known that there is a correspondence between sets and complete, atomic Boolean algebras ($CABA$s) taking a set to its power-set and, conversely, a complete, atomic Boolean algebra to its set of atomic elements. Of course, such a correspondence induces an equivalence between the opposite category of **Set** and the category of $CABA$s.

We modify this result by taking multialgebras over a signature Σ, specifically those whose non-deterministic operations cannot return the empty-set, to $CABA$s with their zero element removed (which we call a *bottomless Boolean algebra*) equipped with a structure of Σ-algebra compatible with its order (that we call *ord-algebras*). Conversely, an ord-algebra over Σ is taken to its set of atomic elements equipped with a structure of multialgebra over Σ. This leads to an equivalence between the category of Σ-multialgebras and the category of ord-algebras over Σ.

The intuition, here, is that if one wishes to do so, non-determinism may be replaced by a sufficiently rich ordering of the underlying structures.

Keywords: multialgebras, ordered algebras, non-deterministic semantics.

Introduction

It is a seminal result (see [24] for a proof) that a correlation between sets and complete, atomic Boolean algebras ($CABA$s) exists: a set is taken to

its power-set, while a *CABA* is taken to its set of atomic elements. These two assignments can be made into functors, giving rise to an equivalence of **Set**^{op} and **CABA**, the category with *CABA*s as objects.

This is part of a broader area of study, known as Stone dualities, which studies relationships between posets and topological spaces and was established by Stone ([22]) and his representation theorem, which states that every Boolean algebra is isomorphic to a field of sets, specifically the algebra of clopen sets of its Stone space (a topological space where points are ultrafilters of the original Boolean algebra). Of course, this corresponds to an equivalence between the category **BA** of Boolean algebras and that of Stone spaces.

In the search of further such equivalences, we focus on a more concrete one, associated to the one between **Set**^{op} and **CABA** in the sense that: we look at an enriched category of sets on one side, namely a category of multialgebras (multialgebras having been originally introduced by Marty in [15] through the notion of hypergroups) over a signature Σ, obtained by adding multioperations to **Set**; and on the other side, at a category attained by equipping the objects of **CABA** with Σ-operations compatible with their orders. And reaching such an equivalence using the aforementioned most general definition of multialgebras on one side, and *CABA*s on the other, is possible: indeed, we do so briefly on Section 5 as a corollary of other of our results. But we choose to focus most of our efforts instead on slightly distinct categories: we are most interested in non-partial multialgebras, where the result of an operation never returns the empty set. Consequently, we exchange *CABA*s for posets corresponding to power-sets with the empty-set removed (that is, *CABA*s without minimum elements, that we call *bottomless Boolean algebras*). This way, a multialgebra, with universe A, is taken to an algebra over the set of non-empty subsets of A, with order given by inclusion and operations given by "accumulating" the operations of the multialgebra, while conversely, a bottomless Boolean algebras is taken to its set of atomic elements, transformed into a multialgebra.

In the area of research of non-deterministic semantics ([2]), specially paraconsistent logics ([7]), this offers an alternative: many logicians are reluctant to appeal to multialgebras in order to characterize a given logic, and the equivalence we here present shows one can, if one chooses to, replace such non-deterministic structures with more classically-behaved algebras, with an added underlying order. Furthermore, using bottomless Boolean

algebras follows a trend: in logic, we are used to considering ordered alge-braic structures without bottoms; for instance, implicative lattices, which are bottomless Heyting algebras. The use of bottomless Boolean algebras feels then justified because the definition of the functor between the cat-egories is much simpler and seems to better correspond to the intuition found in using non-deterministic semantics.

This paper is organized as follows: In the first section, we give the definition of multialgebras we will use and introduce a brief characterization of power-sets without the empty-set. In the second section, we introduce a naive approach to what we would like to accomplish, and show why it fails. In the third section, we introduce the categories for which our desired result actually holds and the functors that will establish an equivalence between them, which we detail in section four. The final section is reserved for related results.

Preliminary versions of this paper can be found in the PhD thesis [23] and in the preprint [11].

1. Preliminary notions

A signature is a collection $\Sigma = \{\Sigma_n\}_{n \in \mathbb{N}}$ of possibly empty, disjoint sets indexed by the natural numbers; when there is no risk of confusion, the union $\bigcup_{n \in \mathbb{N}} \Sigma_n$ will also be denoted by Σ.

A $\Sigma-$multialgebra (also known as multialgebra) is a pair $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma \in \Sigma})$ such that: A is a non-empty set and, for $\sigma \in \Sigma_n$, $\sigma_{\mathcal{A}}$ is a function of the form

$$\sigma_{\mathcal{A}} : A^n \to \mathcal{P}(A) \setminus \{\emptyset\},$$

where $\mathcal{P}(A)$ denotes the power-set of A. If $\sigma_{\mathcal{A}}(\vec{a})$ is a singleton for every $\sigma_{\mathcal{A}}$ and $\vec{a} \in A^n$, then $\mathcal{A}$ is said to be deterministic, and can be identified with a standard algebra.

A homomorphism between $\Sigma-$multialgebras $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma \in \Sigma})$ and $\mathcal{B} = (B, \{\sigma_{\mathcal{B}}\}_{\sigma \in \Sigma})$ is a function $h : A \to B$ satisfying, for any $n \in \mathbb{N}$, $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in A$,

$$\{h(a) : a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\} \subseteq \sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n)).$$

If the inclusion, in the previous equation, were to be replaced by an equal-ity, the resulting h would be a full homomorphism; and a bijective full

homomorphism is called an isomorphism. Whenever h is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$, we write $h : \mathcal{A} \to \mathcal{B}$. If both $\mathcal{A}$ and $\mathcal{B}$ are deterministic, then h can be identified with a standard homomorphism between algebras.

1.1.   Related approaches

Stone-like dualities in particular, and related categorial equivalences in general, are a fertile ground for new results since their conception. Given multialgebras, at least when conceived as relational structures, have permeated mathematics for a long time, it is natural that both concepts have in a way or another interacted in the past.

The simple idea of taking a multialgebra (or a relational structure if one wishes to be more general) and, from that multialgebra, constructing an ordered algebra goes back at least to Dresher and Ore [13]: the constructed algebra is usually referred to as a power algebra, complex, or global algebra. An interesting analysis of this construction is found in Brink [5], which however deals mostly with the broader notion of a relational algebra instead of multialgebras *per se*; a, less studied, alternative to ordered algebras, which attempt to capture inclusion, are the ε-algebras of [8], that in turn try to codify those properties of membership.

One can, however, also find examples of the procedure from which one obtains the power algebra from a multialgebra being applied to the multialgebras we are more interested in, the non-partial ones. Pickett [20], Bruck [6], Walicki and Meldal [25], and Breaz and Pelea [19] do precisely that, although Pickett and Bruck appear to be more preoccupied with the applications of this notion to multigroups and, respectively, multiquasigroups and multigrupoids.

And, despite the fact we use the same construction, not only none of the aforementioned papers delve in the study of Stone-dualities or any related categorical equivalences, they use markedly different definitions of homomorphisms (except Walicki and Meldal, that use no definition of homomorphism as their work is mostly devoted to generalizing identities to the context of multialgebras): more specifically, they use what some logicians call in today's literature full homomorphisms (see [7]); that is not unexpected, since that definition is very appropriate when dealing with the theories of multigroups and hyperrings, but not when dealing with Nmatrices.

Bošnjak and Madarász [4] also use power algebras of non-partial multi-algebras, as long as one considers the obvious connection between the latter and multigraphs, see [9] for a definition of this generalization of graphs (and its applications to logic), and [10] for a few constructions of multialgebras from multigraphs.

In one of the first papers dealing with multialgebras and duality, Hansoul achieves in [14] results very similar to those we wish to obtain, but for different multialgebras: specifically, he uses what some call nowadays partial multialgebras, which are full relational structures; in other words, the result of a multioperation may equal the empty set, and that makes his analysis necessarily very different from ours, non-partial multialgebras being preferred as semantics for non-classical logics (see [2] for the first discussion on the use of non-partial multialgebras as semantic objects, and [3] for the first discussion on the use of partial multialgebras). Nolan, in his thesis ([18]), obtains more general results than Hansoul, taking into his analysis both ordered algebras and Boolean algebras with operators, but also restricts himself to partial multialgebras, giving again preference to full homomorphisms.

To summarize, although the construction leading to power algebras, and multialgebras have interacted before, this was done for the sake of these algebraic structures themselves, or for applications within the realm of pure mathematics, such as in the theory of hypergroups. The difference here lies in the very basic structures we aim to deal with: we are using non-partial multialgebras (that we take the liberty of addressing simply by multialgebras from here forward), together with a weak notion of homomorphism that, although not specially useful for the study of, e.g., hypergroups, is the standard when applying multialgebras to logics. And these profound differences in the category to be analyzed lead to equally profound differences on how the method of producing power algebras behaves, suggesting the use of what we chose to call bottomless Boolean algebras, as we now set out to define.

1.2. Bottomless Boolean algebras

Here, we will understand Boolean algebras mostly as partially-ordered sets (*poset*). A pair $(A, \leq)$ is a Boolean algebra if: $\leq$ is reflexive, anti-symmetric and transitive; there are a maximum (denoted by 1) and a minimum (0), which we shall assume distinct; for every pair of elements $(a, b) \in A^2$, the

set $\{a, b\}$ has a supremum, denoted by $\sup\{a, b\}$ or $a \vee b$, and an infimum, denoted by $\inf\{a, b\}$ or $a \wedge b$; and every element a has a complement b which satisfies

$$b = \min\{c \in A : \sup\{a, c\} = 1\}$$

and

$$b = \max\{c \in A : \inf\{a, c\} = 0\}.$$

A Boolean algebra $(A, \leq)$ is said to be complete if every $S \subseteq A$ has a supremum.

LEMMA 1.1. (i) *Every Boolean algebra $(A, \leq)$ is distributive, meaning*

$$\sup\{a, \inf\{b, c\}\} = \inf\{\sup\{a, b\}, \sup\{a, c\}\}$$

and

$$\inf\{a, \sup\{b, c\}\} = \sup\{\inf\{a, b\}, \inf\{a, c\}\},$$

for any $a, b, c \in A$;

(ii) *every complete Boolean algebra $(A, \leq)$ is infinite distributive, meaning that for any $S \cup \{a\} \subseteq A$, $\sup\{\inf\{a, s\} : s \in S\} = \inf\{a, \sup S\}$ and $\inf\{\sup\{a, s\} : s \in S\} = \sup\{a, \inf S\}$.*

An element a of a Boolean algebra $(A, \leq)$ is said to be an atom if it is minimal in $A \setminus \{0\}$ according to $\leq$, which means that if $b \leq a$, then either $b = 0$ or $b = a$. The set of all the atoms d such that $d \leq a$ will be denoted by A_a. Finally, a Boolean algebra is said to be atomic if, for every one of its elements a, $a = \sup A_a$.

Notice that complete, atomic Boolean algebras are power-sets, at least up to an equivalence (of categories). In one direction, this equivalence takes a Boolean algebra $\mathcal{A} = (A, \leq)$ to the power-set $\mathcal{P}(A_1)$ of the set A_1 of all its atoms, an element $a \in A \setminus \{0\}$ being mapped (by the associated natural isomorphisms of the equivalence) to A_a, and 0 to $\emptyset$. Conversely, a set X its taken by this equivalence to the complete, atomic Boolean algebra that is the power-set of X, $\mathcal{P}(X)$. For more information, look at Theorem 2.4 of [24].

We would like to work with Boolean algebras that are, simultaneously, complete, atomic and bottomless, meaning they lack a bottom element: this seems a contradiction, given we assume Boolean algebras to have bottom elements, but it can be adequately formalized.

DEFINITION 1.2. Given a non-empty partially ordered set $\mathcal{A} = (A, \leq_\mathcal{A})$, we define $\mathcal{A}_0$ as the partially ordered set

$$(A \cup \{0\}, \leq_{\mathcal{A}_0}),$$

where we assume $0 \notin A$, such that $a \leq_{\mathcal{A}_0} b$ if and only if:
   (i) either $a \leq_\mathcal{A} b$;
   (ii) or $a = 0$.

DEFINITION 1.3. A non-empty partially ordered set $\mathcal{A}$ is a *bottomless Boolean algebra* whenever $\mathcal{A}_0$ is a Boolean algebra; if $\mathcal{A}_0$ is a complete or atomic, $\mathcal{A}$ is also said to be complete or atomic, respectively.

Notice that, since $\mathcal{P}(\emptyset)$ only has $\emptyset$ as element, for any complete, atomic and bottomless Boolean algebra $\mathcal{A}$ we cannot have $\mathcal{A}_0 = \mathcal{P}(\emptyset)$, given $\mathcal{A}$ has at least one element and therefore $\mathcal{A}_0$ must have at least two. This means complete, atomic and bottomless Boolean algebras correspond to the power-set of non-empty sets with $\emptyset$ removed.

DEFINITION 1.4. A partially ordered set $(A, \leq)$ with maximum 1 is called semi-complemented when for every $a \in A \setminus \{1\}$ there exists $b \in A$, named a complement of a, such that

$$b = \min\{c \in A : \sup\{a, c\} = 1\}$$

and

$$b = \max\{c \in A : \inf\{a, c\} \text{ does not exist}\}.$$

THEOREM 1.5. *Consider the following properties a partially ordered set* $\mathcal{A} = (A, \leq)$ *may have.*
   (i) *It has a maximum element 1.*
   (ii) $(A, \leq)$ *is semi-complemented.*
   (iii) *All subsets with two elements* $\{a, b\}$ *of A have a supremum.*
   (iii)* *All non-empty subsets S of A have a supremum.*
   (iv) *Denoting by* A_a *the set of minimal elements smaller than a, a =* $\sup A_a$.
   If $\mathcal{A}$ satisfies (i), (ii) *and* (iii), *it is a bottomless Boolean algebra; if $\mathcal{A}$ satisfies* (i), (ii) *and* (iii)*, *it is a complete bottomless Boolean algebra; and if it satisfies* (i), (ii), (iii)* *and* (iv), *it is an atomic, complete Bottomless Boolean algebra.*

PROOF: Suppose that $\mathcal{A} = (A, \leq_{\mathcal{A}})$ is a partially ordered set satisfying (i), (ii) and (iii). Since $\mathcal{A}$ is a partially ordered set, so is $\mathcal{A}_0$ from Definition 1.2. The maximum 1 of $\mathcal{A}$ remains a maximum in $\mathcal{A}_0$, while 0 becomes a minimum. For non-zero elements a and b, the supremum in $\mathcal{A}_0$ of $\{a, b\}$ remains the same as in $\mathcal{A}$, while if $a = 0$ or $b = 0$ the supremum is simply the largest of the two.

If a or b are equal to 0, the infimum is 0, while if $a, b \in A$ there are two cases to consider: if $\inf\{a, b\}$ was defined in $\mathcal{A}$, it remains the same in $\mathcal{A}_0$. If the infimum was not defined in $\mathcal{A}$, it must be 0 in $\mathcal{A}_0$: indeed, $\inf\{a, b\}$ certainly exists in $\mathcal{A}_0$, given that is a Boolean algebra; and if it is not 0, it is in $\mathcal{A}$, being an infimum in this poset as well.

Every element $a \in A \backslash \{1\}$ already has a complement b in $\mathcal{A}$ such that $b = \min\{c \in A : \sup\{a, c\} = 1\}$ and $b = \max\{c \in A : \inf\{a, c\}$ does not exist$\}$. Of course the first equality keeps on holding in $\mathcal{A}_0$, while the second becomes, remembering that the non-defined infima in $\mathcal{A}$ become 0 in $\mathcal{A}_0$,

$$b = \max\{c \in A : \inf\{a, c\} = 0\};$$

the complement of 1 is clearly 0 and vice-versa. This proves that $\mathcal{A}_0$ is a Boolean algebra, and so $\mathcal{A}$ is a bottomless Boolean algebra.

Suppose now $\mathcal{A}$ satisfies instead (i), (ii) and (iii)*: since (iii)* implies (iii), it is clear that $\mathcal{A}$ is a bottomless Boolean algebra; and, since $\mathcal{A}$ is now closed under the suprema of any non-empty sets, and $\sup \emptyset = 0$ in $\mathcal{A}_0$, it is clear that $\mathcal{A}_0$ is closed under any suprema.

Finally, if $\mathcal{A}$ satisfies (i), (ii), (iii)* and (iv), it is to begin with a complete bottomless Boolean algebra; furthermore, clearly $\mathcal{A}_0$ remains atomic, since $\mathcal{A}$ is atomic, what completes the proof that the previous list of conditions imply $\mathcal{A}$ is a complete, atomic and bottomless Boolean algebra.   □

THEOREM 1.6. *The converses of Theorem 1.5 hold, meaning that: a bottomless Boolean algebra satisfies conditions* (i), (ii) *and* (iii)*; a complete bottomless Boolean algebra satisfies conditions* (i), (ii) *and* (iii)*; and an atomic, complete bottomless Boolean algebra satisfies conditions* (i), (ii), (iii)* *and* (iv).

PROOF: Given a partially ordered set $\mathcal{A}$, suppose $\mathcal{A}_0$ is a Boolean algebra.

(i) The maximum 1 of $\mathcal{A}_0$ is still a maximum in $\mathcal{A}$.

(ii) Given any element $a \neq 1$, its complement b in $\mathcal{A}_0$ ends up being also its complement in $\mathcal{A}$. Clearly

$$b = \min\{c \in A : \sup\{a, c\} = 1\}.$$

Now, $\inf\{a, c\}$ does not exist in $\mathcal{A}$ if, and only if, $\inf\{a, c\} = 0$ in $\mathcal{A}_0$: we already proved that if $\inf\{a, c\}$ does not exist in $\mathcal{A}$ then $\inf\{a, c\} = 0$ in $\mathcal{A}_0$, remaining to show the converse; if the infimum of a and c existed in $\mathcal{A}$, it would equal 0 in $\mathcal{A}_0$ given the unicity of the infimum, contradicting that 0 is not in $\mathcal{A}$. This way, we find that in $\mathcal{A}$

$$b = \max\{c \in A : \inf\{a, c\} \text{ does not exist}\},$$

as required.

(iii) The supremum of any set $\{a, b\}$ of cardinality 2 in $\mathcal{A}$ is just its supremum in $\mathcal{A}_0$.

Suppose now $\mathcal{A}_0$ is a complete Boolean algebra.

(iii)* Then the supremum of any non-empty set in $\mathcal{A}$ is its supremum in $\mathcal{A}_0$.

Finally, let $\mathcal{A}_0$ now be an atomic, complete Boolean algebra.

(iv) Clearly $\mathcal{A}_0$ being atomic implies $\mathcal{A}$ being atomic. □

PROPOSITION 1.7. If $(A, \leq_\mathcal{A})$ is a complete, atomic and bottomless Boolean algebra, for any $S \subseteq A$, if

$$S^a = \{s \in S : \inf\{a, s\} \text{ exists}\} \neq \emptyset,$$

then

$$\sup\{\inf\{a, s\} : s \in S^a\} = \inf\{a, \sup S\};$$

if $S^a = \emptyset$, $\inf\{a, \sup S\}$ also does not exist.

PROOF: If $S^a = \emptyset$, this means that $\inf\{a, s\} = 0$ for every $s \in S$ in $\mathcal{A}_0$, and therefore $\inf\{a, \sup S\} = 0$, so that the same infimum no longer exists in $\mathcal{A}$.

If $S^a \neq \emptyset$, all infima and suprema in $\sup\{\inf\{a, s\} : s \in S^a\}$ and $\inf\{a, \sup S\}$ exist in $\mathcal{A}$ and are therefore equal to their counterparts in $\mathcal{A}_0$; given $\sup\{\inf\{a, s\} : s \in S^a\} = \sup\{\inf\{a, s\} : s \in S\}$ in $\mathcal{A}_0$, since $s \in S \setminus S^a$ implies $\inf\{a, s\} = 0$, by the infinite-distributivity of $\mathcal{A}_0$ one proves the desired result. □

The lesson to be taken from this short exposition is that a complete, atomic and bottomless Boolean algebra is a power-set (of a non-empty

set) with the empty-set removed. This will be important to us given our multialgebras cannot return the empty-set as the result of an operation.

1.3.   Tarski algebras and classical implicative lattices

Now, the use of bottomless Boolean algebras may seem an odd choice of structures to take into consideration, given their proximity to Boolean algebras, but there are two important reasons for that choice. First of all, they are very natural when considering the multialgebras, as well as the homomorphisms, typically found when studying non-deterministic semantics. Second, this choice is not as odd as it may appear at first when considering the vast diversity of algebraic structures that are required when dealing with algebraic logic.

We then make a brief comparison of bottomless Boolean algebras with two varieties of algebras, Tarski algebras and classical implicative lattices, both designed to capture the behavior of some negation-free fragment of classical logic: this likeness follows from the fact that, by ignoring the empty-set, we are also, in a sense, considering a positive fragment of something when defining bottomless Boolean algebras. We start with Tarski algebras. In the 1960s, J. Abbott [1] and A. Monteiro [16], with the aim of capturing the implicational fragment of classical logic, independently introduced and studied a class of implication algebras related to Boolean algebras. The latter called them *Tarski algebras* in lectures delivered at Universidad Nacional del Sur (cf. [17]), while the former called them *implicational algebras*. These algebraic structures have only a binary connective $\rightarrow$ and satisfy the following axioms (we use infix notation).

(i) $(x \rightarrow y) \rightarrow x = x$;

(ii) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;

(iii) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$.

Considering our previous commentary, the following result is perhaps not entirely surprising.

THEOREM 1.8. *Given a bottomless Boolean algebra $\mathcal{A} = (A, \leq)$, define*

$$a \rightarrow b = \begin{cases} \sup\{c, b\} & \text{if } a \neq 1, \text{ where } c \text{ is the complement of } a; \\ b & \text{if } a = 1. \end{cases}$$

Then A, equipped with $\rightarrow$, is a Tarski algebra.

PROOF: Although this can be brute-forced, it is easier to see that $a \to b$ is just the implication of the Boolean algebra $\mathcal{A}_0$, restricted to its non-zero elements: indeed, if $a \neq 1$, and c is the complement of a, $c = \neg a$ and $\sup\{c, b\} = \neg c \vee b$; if $a = 1$, $\neg a = 0$, and $b = 0 \vee b = \neg 1 \vee b$. Notice that $\neg a \vee b$ can never be 0, if $b \neq 0$.

As every Boolean algebra is a Taski algebra, we are done. $\square$

Example 1.9. Consider $A = \{a, b, c, 1\}$ and the following table for an implication on A that gives us a structure $\mathcal{A}$.

$\to$	a	b	c	1
a	1	b	c	1
b	a	1	c	1
c	a	b	1	1
1	a	b	c	1

This structure can be shown to be a Tarski algebra, but is not a bottomless Boolean algebra: if it were, $\mathcal{A}_0$ would be a Boolean algebra with 5 elements, what is impossible.

We can show even more: the structure in Example 1.9 is not a classical implicative lattice either. To better explain what that means, a classical implicative lattice, introduced in [12], is an algebra on the signature with symbols $\vee$, $\wedge$ and $\to$ of arity 2, and 1 of arity 0, such that $\vee$, $\wedge$ and 1 make of the structure a lattice with top element 1 (and the usual order, where $x \leq y$ iff $x \vee y = y$), and the following axioms are satisfied:

(i) $x \wedge y \leq z$ iff $y \leq x \to z$;

(ii) $(x \to y) \to x = x$.

As Tarski algebras attempt to capture the implicative fragment of classical propositional logic, classical implicative lattices attempt to capture the positive fragment of the same logic. It is relatively easy to see ([21] being a possibility) that all classical implicative lattices are Tarski algebras. Furthermore, as one can, in a finite classical implicative lattice, obtain a bottom by taking the infimum of all elements, it is also true that all non-trivial finite classical implicative lattices are Boolean algebras.[1]

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[1]By non-trivial we mean with cardinality bigger than one: the only element of a one-element classical implicative lattice is both a top and a bottom, what makes of the one-element classical implicative lattice not a Boolean algebra; the situation changes in a foreseeable way if one wishes to entertain the possibility of a one-element Boolean algebra.

Notice that the structure from Example 1.9 is not a classical implicative lattice because if $\to$ were the implication of a Boolean algebra, $0 \to x$ would equal 1 for every x in $\mathcal{A}$, what clearly is not true whether 0 equals a, b or c.

Now, it is obvious that any Boolean algebra, whether finite or infinite, is a classical implicative lattice. As shown below, the reciprocal is not true.

Example 1.10. Take an infinite set X (say $\mathbb{N}$), and define $A(X)$ as the set of subsets a of X such that a^c is finite, where a^c is the complement $X \setminus a$ of a: these are called the cofinite sets of X.

$A(X)$ has an obvious order, such that $a \leq b$ iff $a \subseteq b$, and a maximal element under this order, X itself. Then $a \vee b = a \cup b$ is the supremum of a and b (and is cofinite since $|(a \cup b)^c| = |a^c \cap b^c| \leq |a^c|$, which is finite), and $a \wedge b = a \cap b$ is the infimum of a and b (and is cofinite since $|(a \cap b)^c| = |a^c \cup b^c| \leq |a^c| + |b^c|$, which is finite since $|a^c|$ and $|b^c|$ are finite).

We then define $a \to b$ as $a^c \cup b$: this is cofinite since $|(a^c \cup b)^c| = |a \cap b^c| \leq |b^c|$, which is finite given that b is cofinite.

(i) if $a \wedge b \leq c$ then $a \cap b \subseteq c$ and so

$$b \subseteq b \cup a^c = X \cap (b \cup a^c) = (a \cup a^c) \cap (b \cup a^c) = (a \cap b) \cup a^c \subseteq c \cup a^c,$$

that is, $b \leq a \to c$. Conversely, $b \leq a \to c$ implies that $b \subseteq c \cup a^c$, hence

$$a \cap b \subseteq a \cap (c \cup a^c) = (a \cap c) \cup (a \cap a^c) = a \cap c \subseteq c,$$

that is, $a \wedge b \leq c$.

(ii) $(a \to b) \to a$ equals

$$(a^c \cup b)^c \cup a = (a \cap b^c) \cup a = (a \cup a) \cap (b^c \cup a) = a \cap (b^c \cup a) = a.$$

So, we have proven $A(X)$ is a classical implicative lattice. But it cannot be a Boolean algebra: no element a of $A(X)$ can be a bottom, since removing a single element of X from a gives an element of $A(X)$ strictly smaller than a itself.

There is, however, a more involved, although also more natural, example of a classical implicative lattice that is not a Boolean algebra: the Lindenbaum-Tarski algebra of the positive fragment of classical logic in an infinite number of variables. That is, in fact, the very reason why classical implicative lattices were defined, to model the properties of these fragments.

Finally, to completely characterize the relationship between Tarski, Boolean and Bottomless Boolean algebras, and classical implicative lattices, we just need to prove that no non-trivial Bottomless Boolean algebra is a classical implicative lattice: of course, the one-element bottomless Boolean algebra is also a classical implicative lattice. Suppose, then, that the poset $\mathcal{A} = (A, \leq)$ is a bottomless Boolean algebra, a classical implicative lattice, and has more than one element in its domain, so $\mathcal{A}_0$ is a Boolean algebra with at least four elements. There is, therefore, an element a in $\mathcal{A}_0$ that is neither 0 nor 1, and so is $\neg a$: both a and $\neg a$ must then be in $\mathcal{A}$, and so must $0 = a \wedge \neg a$ since $\mathcal{A}$ is a classical implicative lattice. This is a contradiction, given that $\mathcal{A}$, as a bottomless Boolean algebra with at least two elements, cannot have a bottom.

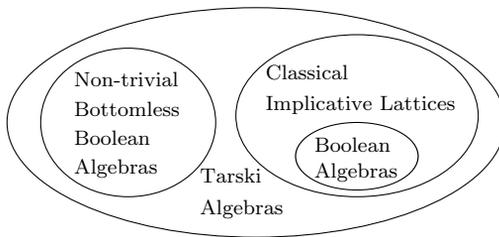We therefore reach the characterization shown in Figure 1.

Figure 1. Several classes of Boolean-like algebras

2. A naive approach to an equivalence of categories

Consider the categories **Alg**(Σ) of Σ-algebras, with homomorphisms between Σ-algebras as morphisms, and **MAlg**(Σ) of Σ-multialgebras, with homomorphisms between Σ-multialgebras as morphisms, and with composition of morphisms and identity morphisms as in **Set**.

For simplicity, denote the set of non-empty subsets of A, $\mathcal{P}(A) \setminus \{\emptyset\}$, by $\mathcal{P}^*(A)$. For a Σ-multialgebra $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma \in \Sigma})$, consider the Σ-algebra $\mathsf{P}(\mathcal{A}) = (\mathcal{P}^*(A), \{\sigma_{\mathsf{P}(\mathcal{A})}\}_{\sigma \in \Sigma})$ where, for a $\sigma \in \Sigma_n$ and nonempty $A_1, \ldots, A_n \subseteq A$,

$$\sigma_{\mathsf{P}(\mathcal{A})}(A_1, \ldots, A_n) = \bigcup_{(a_1,\ldots,a_n) \in A_1 \times \cdots \times A_n} \sigma_{\mathcal{A}}(a_1, \ldots, a_n).$$

Again, for simplicity, we may write the previous equation as $\sigma_{\mathsf{P}(\mathcal{A})}(A_1, \ldots, A_n) = \bigcup\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) : a_i \in A_i\}$. We also define, for $\mathcal{A}$ and $\mathcal{B}$ two Σ-multialgebras, and a homomorphism $h : \mathcal{A} \to \mathcal{B}$, the function $\mathsf{P}(h) : \mathsf{P}(A) \to \mathsf{P}(B)$ such that, for every $\emptyset \neq A' \subseteq A$,

$$\mathsf{P}(h)(A') = \{h(a) : a \in A'\}.$$

One could hope that $\mathsf{P}(h)$ is actually a Σ-homomorphism, perhaps making of P a functor from $\mathbf{MAlg}(\Sigma)$ to $\mathbf{Alg}(\Sigma)$, but the following result shows this is usually not the case.

LEMMA 2.1. *For $\mathcal{A}$ and $\mathcal{B}$ two Σ-multialgebras and $h : \mathcal{A} \to \mathcal{B}$ a homomorphism, $\mathsf{P}(h)$ satisfies*

$$\mathsf{P}(h)(\sigma_{\mathsf{P}(\mathsf{A})}(A_1, \ldots, A_n)) \subseteq \sigma_{\mathsf{P}(\mathsf{B})}(\mathsf{P}(h)(A_1), \ldots, \mathsf{P}(h)(A_n))$$

for all $\sigma \in \Sigma$ and nonempty $A_1, \ldots, A_n \subseteq A$. If h is a full homomorphism, $\mathsf{P}(h)$ is a homomorphism.

PROOF: Given $\sigma \in \Sigma_n$ and nonempty $A_1, \ldots, A_n \subseteq A$, we have that

$$\sigma_{\mathsf{P}(\mathcal{B})}(\mathsf{P}(h)(A_1), \ldots, \mathsf{P}(h)(A_n)) = \bigcup\{\sigma_{\mathcal{B}}(b_1, \ldots, b_n) : b_i \in \mathsf{P}(h)(A_i)\} =$$

$$\bigcup\{\sigma_{\mathcal{B}}(b_1, \ldots, b_n) : b_i \in \{h(a) : a \in A_i\}\} =$$

$$\bigcup\{\sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n)) : a_i \in A_i\} \supseteq$$

$$\bigcup\{\{h(a) : a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\} : a_i \in A_i\} =$$

$$\{h(a) : a \in \bigcup\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) : a_i \in A_i\}\} =$$

$$\{h(a) : a \in \sigma_{\mathsf{P}(\mathcal{A})}(A_1, \ldots, A_n)\} = \mathsf{P}(h)(\sigma_{\mathsf{P}(\mathcal{A})}(A_1, \ldots, A_n)),$$
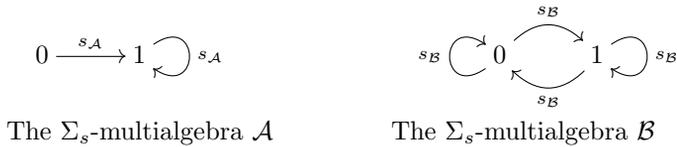
so that $\mathsf{P}(h)$ satisfies the required property.

If h is a full homomorphism, $\sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n)) = \{h(a) : a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\}$, and the inclusions above become identities. $\qquad\square$

So, let us restrict P for a moment to the category $\mathbf{MAlg}_=(\Sigma)$, of Σ-multialgebras with only full homomorphisms between them as morphisms, and let us call this new transformation $\mathsf{P}_= : \mathbf{MAlg}_=(\Sigma) \to \mathbf{Alg}(\Sigma)$.
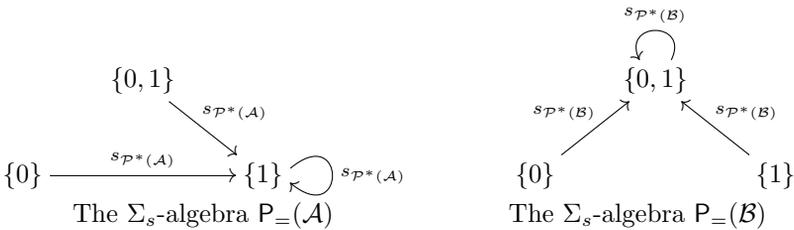
PROPOSITION 2.2. $\mathsf{P}_=$ is a functor from $\mathbf{MAlg}_=(\Sigma)$ to $\mathbf{Alg}(\Sigma)$.

Unfortunately, $\mathsf{P}_=$ is not injective on objects: take the signature Σ_s with a single unary operator s, and consider the Σ_s-multialgebras $\mathcal{A} = (\{0,1\}, \{s_\mathcal{A}\})$ and $\mathcal{B} = (\{0,1\}, \{s_\mathcal{B}\})$ such that: $s_\mathcal{A}(0) = s_\mathcal{A}(1) = \{1\}$ and $s_\mathcal{B}(0) = s_\mathcal{B}(1) = \{0,1\}$.

$$0 \xrightarrow{\ s_\mathcal{A}\ } 1 \circlearrowleft s_\mathcal{A} \qquad\qquad s_\mathcal{B} \circlearrowright 0 \underset{s_\mathcal{B}}{\overset{s_\mathcal{B}}{\rightleftarrows}} 1 \circlearrowleft s_\mathcal{B}$$

The Σ_s-multialgebra $\mathcal{A}$ $\qquad\qquad$ The Σ_s-multialgebra $\mathcal{B}$

Clearly the two of them are not isomorphic, given that the result of an operation in $\mathcal{A}$ always has cardinality 1 and in $\mathcal{B}$ alway has cardinality 2.

However, we have that $s_{\mathsf{P}_=(\mathcal{A})}(\{0\}) = s_{\mathsf{P}_=(\mathcal{A})}(\{1\}) = s_{\mathsf{P}_=(\mathcal{A})}(\{0,1\}) = \{1\}$, while $s_{\mathsf{P}_=(\mathcal{B})}(\{0\}) = s_{\mathsf{P}_=(\mathcal{A})}(\{1\}) = s_{\mathsf{P}_=(\mathcal{A})}(\{0,1\}) = \{0,1\}$.

$$\{0,1\} \qquad\qquad\qquad\qquad \{0,1\} \circlearrowright^{s_{\mathcal{P}^*(\mathcal{B})}}$$

The Σ_s-algebra $\mathsf{P}_=(\mathcal{A})$ $\qquad\qquad$ The Σ_s-algebra $\mathsf{P}_=(\mathcal{B})$

Taking the function $h : \mathcal{P}^*(A) \to \mathcal{P}^*(B)$ such that $h(\{0\}) = \{0\}$, $h(\{1\}) = \{0,1\}$, and $h(\{0,1\}) = \{1\}$, we see that it is a bijection and a homomorphism, and therefore $h : \mathsf{P}_=(\mathcal{A}) \to \mathsf{P}_=(\mathcal{B})$ is an isomorphism.

3. A solution: ord-algebras

The problem with our definition of $\mathsf{P}_=$ is that it disregards the structure of the universe of $\mathcal{P}(\mathcal{A})$. So, we change our target category to reflect this structure.

3.1. The category OAlg(Σ), and the functor $\mathbb{P}$

DEFINITION 3.1. Given a signature Σ, an ord-algebra over Σ is a triple $\mathcal{A} = (A, \{\sigma_\mathcal{A}\}_{\sigma\in\Sigma}, \leq_\mathcal{A})$ such that:

(i) $(A, \{\sigma_\mathcal{A}\}_{\sigma\in\Sigma})$ is a Σ-algebra;

(ii) $(A, \leq_\mathcal{A})$ is a complete, atomic and bottomless Boolean algebra;

(iii) if A_a is the set of minimal elements of $(A, \leq_\mathcal{A})$ (atoms) less than or equal to a, for all $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n$ we have that

$$\sigma_\mathcal{A}(a_1, \ldots, a_n) = \sup\{\sigma_\mathcal{A}(b_1, \ldots, b_n) : (b_1, \ldots, b_n) \in A_{a_1} \times \cdots \times A_{a_n}\}.$$

Here, it should be clear that ord-algebras are a class of ordered algebras that aim to capture precisely those properties of the power algebras of those multialgebras which interest us: the non-partial ones. This shall be formalized further ahead.

PROPOSITION 3.2. Let $\mathcal{A}$ be an ord-algebra over Σ, $\sigma \in \Sigma_n$ and a_1 through a_n, and b_1 through b_n in A such that $a_1 \leq_\mathcal{A} b_1$, $\ldots$, $a_n \leq_\mathcal{A} b_n$. Then, $\sigma_\mathcal{A}(a_1, \ldots, a_n) \leq_\mathcal{A} \sigma_\mathcal{A}(b_1, \ldots, b_n)$.

PROOF: Since, for every $i \in \{1, \ldots, n\}$, $a_i \leq_\mathcal{A} b_i$, we have that $A_{a_i} \subseteq A_{b_i}$, one concludes that $A_{a_1} \times \cdots \times A_{a_n} \subseteq A_{b_1} \times \cdots \times A_{b_n}$; this way,

$$\sigma_\mathcal{A}(a_1, \ldots, a_n) = \sup\{\sigma_\mathcal{A}(c_1, \ldots, c_n) : c_i \in A_{a_i}\} \leq_\mathcal{A}$$

$$\sup\{\sigma_\mathcal{A}(c_1, \ldots, c_n) : c_i \in A_{b_i}\} = \sigma_\mathcal{A}(b_1, \ldots, b_n). \qquad \square$$

For a Σ-multialgebra $\mathcal{A} = (A, \{\sigma_\mathcal{A}\}_{\sigma\in\sigma})$, we define $\mathbb{P}(\mathcal{A})$ as the ord-algebra

$$(\mathcal{P}^*(A), \{\sigma_{\mathbb{P}(\mathcal{A})}\}_{\sigma\in\Sigma}, \leq_{\mathbb{P}(\mathcal{A})})$$

over Σ such that $(\mathcal{P}^*(A), \{\sigma_{\mathbb{P}(\mathcal{A})}\}_{\sigma\in\Sigma})$ is exactly the Σ-algebra $\mathsf{P}(\mathcal{A})$ defined at the beginning of Section 2 and, for nonempty subsets A_1 and A_2 of A, $A_1 \leq_{\mathbb{P}(\mathcal{A})} A_2$ if and only if $A_1 \subseteq A_2$. Since:

(i) $\mathsf{P}(\mathcal{A})$ is a Σ-algebra;

(ii) $(\mathcal{P}^*(A), \leq_{\mathbb{P}(\mathcal{A})})$ is a complete, atomic and bottomless Boolean algebra, given that $\mathcal{P}(A)$ is a complete, atomic Boolean algebra with at least two elements;

(iii) and, for $\sigma \in \Sigma_n$ and $\emptyset \neq A_1, \ldots, A_n \subseteq A$, since the atoms below A_i are exactly $A_{A_i} = \{\{a\} : a \in A_i\}$,

$$\sigma_{\mathbb{P}(\mathcal{A})}(A_1, \ldots, A_n) = \bigcup\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) : a_i \in A_i\} =$$

$$\bigcup\{\sigma_{\mathbb{P}(\mathcal{A})}(\{a_1\}, \ldots, \{a_n\}) : \{a_i\} \in A_{A_i}\};$$

we indeed have that $\mathbb{P}(\mathcal{A})$ is an ord-algebra over Σ.

DEFINITION 3.3. Let $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma \in \Sigma}, \leq_{\mathcal{A}})$ and $\mathcal{B} = (B, \{\sigma_{\mathcal{B}}\}_{\sigma \in \Sigma}, \leq_{\mathcal{B}})$ be two ord-algebras over Σ. A function $h : A \to B$ is said to be a homomorphism of ord-algebras, in which case we write $h : \mathcal{A} \to \mathcal{B}$, when:

(i) for all $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in A$ we have that

$$h(\sigma_{\mathcal{A}}(a_1, \ldots, a_n)) \leq_{\mathcal{B}} \sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n));$$

(ii) h is continuous, meaning that, for every non-empty subset $A' \subseteq A$,

$$h(\sup A') = \sup\{h(a) : a \in A'\};$$

(iii) h maps minimal elements of $(A, \leq_{\mathcal{A}})$ to minimal elements of $(B, \leq_{\mathcal{B}})$.

Notice that a homomorphism of ord-algebras is essentially an "almost Σ-homomorphism" which is also continuous and minimal-elements-preserving. Notice also that a homomorphism of ord-algebras is order preserving: if $a \leq_{\mathcal{A}} b$, then $b = \sup\{a, b\}$, and therefore $h(b) = \sup\{h(a), h(b)\}$, meaning that $h(a) \leq_{\mathcal{B}} h(b)$.

LEMMA 3.4. *The composition of homomorphisms of ord-algebras returns a homomorphism of ord-algebras.*

PROOF: Take ord-algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ over Σ, and homomorphisms of ord-algebras $h : \mathcal{A} \to \mathcal{B}$ and $h' : \mathcal{B} \to \mathcal{C}$.

(i) $h' \circ h$ obviously is a function from A to C, so let $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in A$: we have that, since both h' and h are homomorphisms of ord-algebras,

$$h' \circ h(\sigma_{\mathcal{A}}(a_1, \ldots, a_n)) = h'(h(\sigma_{\mathcal{A}}(a_1, \ldots, a_n))) \leq_{\mathcal{C}} h'(\sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n))),$$

because h' is order-preserving and

$$h(\sigma_{\mathcal{A}}(a_1, \ldots, a_n)) \leq_{\mathcal{B}} \sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n));$$

and

$$h'(\sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n))) \leq_{\mathcal{C}} \sigma_{\mathcal{C}}(h'(h(a_1)), \ldots, h'(h(a_n))) =$$

$$\sigma_{\mathcal{C}}(h' \circ h(a_1), \ldots, h' \circ h(a_n))$$

since h' is an "almost homomorphism".

(ii) Given a non-empty $A' \subseteq A$, we have that $h(\sup A') = \sup\{h(a) : a \in A'\}$ and, denoting $\{h(a) : a \in A'\}$ as B', we have that $h'(\sup B') = \sup\{h'(b) : b \in B'\}$; since $\sup B' = h(\sup A')$, we obtain

$$h' \circ h(\sup A') = \sup\{h'(b) : b \in B'\} = \sup\{h' \circ h(a) : a \in A'\},$$

which means that $h' \circ h$ is continuous.

(iii) Finally, if $a \in A$ is a minimal element, $h(a) \in B$ is a minimal element, since h preserves minimal elements, and for the same reason $h' \circ h(a) = h'(h(a)) \in C$ remains a minimal element still, and from all of the above $h' \circ h$ is a homomorphism of ord-algebras. $\qquad\square$

PROPOSITION 3.5. When we take as objects all ord-algebras over Σ and as morphisms all the homomorphisms of ord-algebras between them, with composition of morphisms and identity morphisms as in **Set**, the resulting structure is a category, denoted by **OAlg**(Σ).

THEOREM 3.6. *The transformation taking a Σ-multialgebra $\mathcal{A}$ to $\mathbb{P}(\mathcal{A})$, and a homomorphism $h : \mathcal{A} \to \mathcal{B}$ to the homomorphism $\mathbb{P}(h) : \mathbb{P}(\mathcal{A}) \to \mathbb{P}(\mathcal{B})$ of ord-algebras such that, for every $\emptyset \neq A' \subseteq A$,*

$$\mathbb{P}(h)(A') = \{h(a) : a \in A'\},$$

is a functor, of the form $\mathbb{P} : \textbf{MAlg}(\Sigma) \to \textbf{OAlg}(\Sigma)$.

PROOF: First we must show that $\mathbb{P}(h)$ is, in fact, a homomorphism of ord-algebras: given Lemma 2.1 and the fact that $\mathsf{P}(h) = \mathbb{P}(h)$, we have that $\mathbb{P}(h)$ satisfies the first condition for being a homomorphism of ord-algebras; and, if $\emptyset \neq A''$ is a subset of $\mathcal{P}(A)$, we have that

$$\mathbb{P}(h)(\sup A'') = \{h(a) : a \in \sup A''\} = \{h(a) : a \in \bigcup A''\} =$$

$$\bigcup\{\{h(a) : a \in A'\} : A' \in A''\} = \bigcup\{\mathbb{P}(h)(A') : A' \in A''\} =$$

$$\sup\{\mathbb{P}(h)(A') : A' \in A''\},$$

what proves the satisfaction of the second condition; for the third condition, we remember that the minimal elements of $(\mathcal{P}^*(A), \subseteq)$ are the singletons, that is, sets of the form $\{a\}$ with $a \in A$, and since $\mathbb{P}(h)(\{a\}) = \{h(a)\}$, $\mathbb{P}(h)$ preserves minimal elements.

If $h : \mathcal{A} \to \mathcal{A}$ is the identity $id_{\mathcal{A}}$ of $\mathcal{A}$, then

$$\mathbb{P}(id_{\mathcal{A}})(A') = \{id_{\mathcal{A}}(a) : a \in A'\} = \{a : a \in A'\} = A',$$

meaning $\mathbb{P}(id_{\mathcal{A}})$ is again the identity. Finally, if $h : \mathcal{A} \to \mathcal{B}$ and $h' : \mathcal{B} \to \mathcal{C}$ are homomorphisms of multialgebras,

$$\mathbb{P}(h' \circ h)(A') = \{h' \circ h(a) : a \in A'\} =$$

$$\{h'(b) : b \in \mathbb{P}(h)(A')\} = \mathbb{P}(h') \circ \mathbb{P}(h)(A'),$$

and thus $\mathbb{P}$ is indeed a functor. □

Here, we start to better understand the role played by power algebras: if $\mathcal{A}$ is a multialgebra, $\mathbb{P}(\mathcal{A})$ is a certain power algebra of $\mathcal{A}$; specifically, the one presented conveniently as a bottomless Boolean algebra.

3.2. $\mathbb{P}$ may be seen as part of a monad

As is the case with the power-set functor, from **Set** to itself, we may see $\mathbb{P}$, or even P and P$_=$, as being part of a monad, although some minor modifications are necessary. So, consider the endofunctor $\tilde{\mathbb{P}} : \mathbf{MAlg}(\Sigma) \to \mathbf{MAlg}(\Sigma)$ such that, for a Σ-multialgebra $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma \in \Sigma})$, $\tilde{\mathbb{P}}\mathcal{A}$ is the Σ-multialgebra with universe $\mathcal{P}^*(A)$ and operations given by

$$\sigma_{\tilde{\mathbb{P}}\mathcal{A}}(A_1, \ldots, A_n) = \{\{a\} \in \mathcal{P}^*(A) : a \in \bigcup\{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) : a_i \in A_i\}\},$$

for σ an n-ary symbol and A_1 through A_n non-empty subsets of A; and for Σ-multialgebras $\mathcal{A}$ and $\mathcal{B}$, a homomorphism $h : \mathcal{A} \to \mathcal{B}$ and a non-empty $A' \subseteq A$, $\tilde{\mathbb{P}}h : \tilde{\mathbb{P}}\mathcal{A} \to \tilde{\mathbb{P}}\mathcal{B}$ satisfying $\tilde{\mathbb{P}}h(A') = \{h(a) : a \in A'\}$ (we omit a pair of parenthesis in this expression only for this section, given it is heavy in compositions of functors). Notice that $\tilde{\mathbb{P}}\mathcal{A}$ is almost the same as P$(\mathcal{A})$, with the difference that in the latter, operations return subsets of A, while in the former they return sets of singletons of A, whose union is exactly the result of the operation as performed in P$(\mathcal{A})$.

For the natural transformations to form a monad together with $\tilde{\mathbb{P}}$, we chose the obvious candidates: $\eta : 1_{\mathbf{MAlg}(\Sigma)} \to \tilde{\mathbb{P}}$ and $\epsilon : \tilde{\mathbb{P}} \circ \tilde{\mathbb{P}} \to \tilde{\mathbb{P}}$ given by, for a Σ-multialgebra $\mathcal{A}$, an element a of $\mathcal{A}$ and a non-empty collection $\{A_i\}_{i \in I}$ of non-empty subsets of A, $\eta_{\mathcal{A}}(a) = \{a\}$ and $\epsilon_{\mathcal{A}}(\{A_i\}_{i \in I}) = \bigcup\{A_i : i \in I\}$.

PROPOSITION 3.7. For any Σ-multialgebra $\mathcal{A}$, $\eta_{\mathcal{A}}$ and $\epsilon_{\mathcal{A}}$ are homomorphisms.

PROPOSITION 3.8. For any Σ-multialgebras $\mathcal{A}$ and $\mathcal{B}$, and homomorphism $h : \mathcal{A} \to \mathcal{B}$, the identities $\tilde{\mathbb{P}}h \circ \eta_{\mathcal{A}} = \eta_{\mathcal{B}} \circ h$ and $\tilde{\mathbb{P}}h \circ \epsilon_{\mathcal{A}} = \epsilon_{\mathcal{B}} \circ \tilde{\mathbb{P}}\tilde{\mathbb{P}}h$ are satisfied, meaning that η and ϵ are natural transformations.

PROOF: Let a be an element of $\mathcal{A}$. We have that $\tilde{\mathbb{P}}h \circ \eta_{\mathcal{A}}(a) = \tilde{\mathbb{P}}h(\eta_{\mathcal{A}}(a))$, and since $\eta_{\mathcal{A}}(a) = \{a\}$, we have that $\tilde{\mathbb{P}}h \circ \eta_{\mathcal{A}}(a) = \{h(a)\}$. Meanwhile, $\eta_{\mathcal{B}} \circ h(a) = \eta_{\mathcal{A}}(h(a)) = \{h(a)\}$, and as stated both expressions coincide.

Now, let $\{A_i\}_{i \in I}$ be an element of $\tilde{\mathbb{P}}\tilde{\mathbb{P}}\mathcal{A}$, meaning it is a non-empty set of non-empty subsets of $\mathcal{A}$: $\tilde{\mathbb{P}}h \circ \epsilon_{\mathcal{A}}(\{A_i\}_{i \in I}) = \tilde{\mathbb{P}}h(\epsilon_{\mathcal{A}}(\{A_i\}_{i \in I}))$, and since $\epsilon_{\mathcal{A}}(\{A_i\}_{i \in I}) = \bigcup\{A_i : i \in I\}$, the whole expression simplifies to $\{h(a) : a \in \bigcup\{A_i : i \in I\}\}$. In turn,

$$\epsilon_{\mathcal{B}} \circ \tilde{\mathbb{P}}\tilde{\mathbb{P}}h(\{A_i\}_{i \in I}) = \epsilon_{\mathcal{B}}(\{\{h(a) : a \in A_i\} : i \in I\}),$$

which is equal to

$$\bigcup\{\{h(a) : a \in A_i\}i \in I\} = \{h(a) : a \in \bigcup\{A_i : i \in I\}\},$$

giving us the desired equality.   □

THEOREM 3.9. *The triple of* $\tilde{\mathbb{P}}$, η *and* ϵ *forms a monad.*

PROOF: Let $\mathcal{A}$ be a Σ-multialgebra. We first must prove $\epsilon \circ \tilde{\mathbb{P}}\epsilon = \epsilon \circ \epsilon\tilde{\mathbb{P}}$, what amounts to $\epsilon_{\mathcal{A}} \circ \tilde{\mathbb{P}}\epsilon_{\mathcal{A}} = \epsilon_{\mathcal{A}} \circ \epsilon_{\tilde{\mathbb{P}}\mathcal{A}}$, as homomorphisms from $\tilde{\mathbb{P}}^3\mathcal{A}$ to $\tilde{\mathbb{P}}\mathcal{A}$. So, let $\{\{A_i^j\}_{i \in I}\}_{j \in J}$ be an element of $\tilde{\mathbb{P}}^3\mathcal{A}$, where I and J are non-empty sets of indexes and all A_i^j are non-empty subsets of A:

$$\epsilon_{\mathcal{A}} \circ \tilde{\mathbb{P}}\epsilon_{\mathcal{A}}(\{\{A_i^j\}_{i \in I}\}_{j \in J}) = \epsilon_{\mathcal{A}}(\{\epsilon_{\mathcal{A}}(\{A_i^j : i \in I\}) : j \in J\}) =$$

$$\epsilon_{\mathcal{A}}(\{\bigcup\{A_i^j : i \in I\} : j \in J\}),$$

what equals $\bigcup\{\bigcup\{A_i^j : i \in I\} : j \in J\}$, while $\epsilon_{\mathcal{A}} \circ \epsilon_{\tilde{\mathbb{P}}\mathcal{A}}(\{\{A_i^j\}_{i \in I}\}_{j \in J}) = \epsilon_{\mathcal{A}}(\bigcup\{\{A_i^j : j \in J\}\}_{i \in I}) = \bigcup\{\bigcup\{A_i^j : j \in J\} : i \in I\}$, and it is clear that both sets are the same.

It remains to be proven $\epsilon \circ \tilde{\mathbb{P}}\eta = \epsilon \circ \eta\tilde{\mathbb{P}} = 1_{\tilde{\mathbb{P}}}$, meaning that $\epsilon_{\mathcal{A}} \circ \eta_{\tilde{\mathbb{P}}\mathcal{A}} = \epsilon_{\mathcal{A}} \circ \tilde{\mathbb{P}}\eta_{\mathcal{A}}$, as homomorphisms from $\tilde{\mathbb{P}}\mathcal{A}$ to $\tilde{\mathbb{P}}\mathcal{A}$, and this coincides with the identity homomorphism on this multialgebra as well. So, we take a non-empty subset A' of A, and we have that $\epsilon_{\mathcal{A}} \circ \eta_{\tilde{\mathbb{P}}\mathcal{A}}(A') = \epsilon_{\mathcal{A}}(\{A'\}) = A'$, while for the other expression one derives

$$\epsilon_{\mathcal{A}} \circ P\eta_{\mathcal{A}}(A') = \epsilon_{\mathcal{A}}(\{\eta_{\mathcal{A}}(a) \ : \ a \in A'\}) = \epsilon_{\mathcal{A}}(\{\{a\} \ : \ a \in A'\}) =$$

$$\bigcup\{\{a\} : a \in A'\} = A',$$

what completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.3. Multialgebras of atoms

Given an ord-algebra $\mathcal{A}$ over Σ, take the set $\mathbb{A}((A, \leq_{\mathcal{A}}))$ of atoms of $(A, \leq_{\mathcal{A}})$, that is, the set of minimal elements of this partially ordered set (equal to A_1 as well). For a $\sigma \in \Sigma_n$ and atoms $a_1, \ldots, a_n \in \mathbb{A}((A, \leq_{\mathcal{A}}))$, we define

$$\sigma_{\mathbb{A}(\mathcal{A})}(a_1, \ldots, a_n) = \{a \in \mathbb{A}((A, \leq_{\mathcal{A}})) : a \leq_{\mathcal{A}} \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\} = A_{\sigma_{\mathcal{A}}(a_1, \ldots, a_n)}.$$

This way, $(\mathbb{A}((A, \leq_{\mathcal{A}})), \{\sigma_{\mathbb{A}(\mathcal{A})}\}_{\sigma \in \Sigma})$ becomes a Σ-multialgebra, that we will denote by $\mathbb{A}(\mathcal{A})$ and call the multialgebra of atoms of $\mathcal{A}$. Given ord-algebras $\mathcal{A}$ and $\mathcal{B}$ over Σ, and a homomorphism of ord-algebras $h : \mathcal{A} \to \mathcal{B}$, we also define

$$\mathbb{A}(h) : \mathbb{A}((A, \leq_{\mathcal{A}})) \to \mathbb{A}((B, \leq_{\mathcal{B}}))$$

as the restriction of h to $\mathbb{A}((A, \leq_{\mathcal{A}})) \subseteq A$. It is well-defined since every homomorphism of ord-algebras preserves minimal elements, that is, atoms.

For $\sigma \in \Sigma_n$ and atoms $a_1, \ldots, a_n \in \mathbb{A}((A, \leq_{\mathcal{A}}))$ we have that

$$\{\mathbb{A}(h)(a) : a \in \sigma_{\mathbb{A}(\mathcal{A})}(a_1, \ldots, a_n)\} = \{h(a) : a \in \sigma_{\mathbb{A}(\mathcal{A})}(a_1, \ldots, a_n)\} =$$

$$\{h(a) \in \mathbb{A}((B, \leq_{\mathcal{B}})) : a \leq_{\mathcal{A}} \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\}$$

and, since $a \leq_{\mathcal{A}} \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$ implies $h(a) \leq_{\mathcal{B}} h(\sigma_{\mathcal{A}}(a_1, \ldots, a_n))$ given that h is order preserving, which in turn implies that $h(a) \leq_{\mathcal{B}}$

$\sigma_\mathcal{B}(h(a_1), \ldots, h(a_n))$ since h is an "almost homomorphism", we get that

$$\{h(a) \in \mathbb{A}((B, \leq_\mathcal{B})) : a \leq_\mathcal{A} \sigma_\mathcal{A}(a_1, \ldots, a_n)\} \subseteq$$

$$\{b \in \mathbb{A}((B, \leq_\mathcal{B})) : b \leq_\mathcal{B} \sigma_\mathcal{B}(h(a_1), \ldots, h(a_n)\} =$$

$$\sigma_{\mathbb{A}(\mathcal{B})}(h(a_1), \ldots, h(a_n)) = \sigma_{\mathbb{A}(\mathcal{B})}(\mathbb{A}(h)(a_1), \ldots, \mathbb{A}(h)(a_n)),$$

what proves that $\mathbb{A}(h)$ is a homomorphism between Σ-multialgebras, and we may write $\mathbb{A}(h) : \mathbb{A}(\mathcal{A}) \to \mathbb{A}(\mathcal{B})$.

The natural question is if $\mathbb{A} : \mathbf{OAlg}(\Sigma) \to \mathbf{MAlg}(\Sigma)$ is a functor, to which the answer is yes: it is easy to see that it distributes over the composition of morphisms and preserves the identity ones.

4. OAlg(Σ) and MAlg(Σ) are equivalent

Now, we aim to prove that $\mathbf{OAlg}(\Sigma)$ and $\mathbf{MAlg}(\Sigma)$ are actually equivalent categories, the equivalence being given by the functors $\mathbb{P}$ and $\mathbb{A}$. In order to prove that $\mathbb{P}$ and $\mathbb{A}$ form an equivalence of categories it is enough to prove that both are full and faithful and $\mathbb{A}$ is a right adjoint of $\mathbb{P}$.

4.1. $\mathbb{P}$ and $\mathbb{A}$ are full and faithful

PROPOSITION 4.1. $\mathbb{P}$ is faithful.

PROOF: Given Σ-multialgebras $\mathcal{A}$ and $\mathcal{B}$, and homomorphisms $h, h' : \mathcal{A} \to \mathcal{B}$, if $\mathbb{P}(h) = \mathbb{P}(h')$, we have that, for every $a \in A$,

$$\{h(a)\} = \mathbb{P}(h)(\{a\}) = \mathbb{P}(h')(\{a\}) = \{h'(a)\},$$

and therefore $h = h'$. □

PROPOSITION 4.2. $\mathbb{A}$ is faithful.

PROOF: Given ord-algebras $\mathcal{A}$ and $\mathcal{B}$ over Σ, and homomorphisms of ord-algebras $h, h' : \mathcal{A} \to \mathcal{B}$, suppose that $\mathbb{A}(h) = \mathbb{A}(h')$. Then, for every $a \in A$, we can write $a = \sup A_a$, since $(A, \leq_\mathcal{A})$ is atomic.

Since h and h' are continuous, $h(a) = \sup\{h(a') : a' \in A_a\}$ and $h'(a) = \sup\{h'(a') : a' \in A_a\}$. But, since $\mathbb{A}(h) = \mathbb{A}(h')$, h and h' are the same when

restricted to atoms, and therefore $\{h(a') : a' \in A_a\} = \{h'(a') : a' \in A_a\}$. This means that $h(a) = h'(a)$ and, since a is arbitrary, $h = h'$. $\qquad\square$

THEOREM 4.3. $\mathbb{P}$ *is full.*

PROOF: Given Σ-multialgebras $\mathcal{A}$ and $\mathcal{B}$, and a homomorphism of ord-algebras $h : \mathbb{P}(\mathcal{A}) \to \mathbb{P}(\mathcal{B})$, to prove that $\mathbb{P}$ is full we must find a homomorphism $h' : \mathcal{A} \to \mathcal{B}$ such that $\mathbb{P}(h') = h$.

For every $a \in A$, $\{a\}$ is an atom and, since h preserves atoms, $h(\{a\})$ is an atom of $\mathbb{P}(B)$, and therefore of the form $\{b_a\}$ for some $b_a \in B$. We define $h' : \mathcal{A} \to \mathcal{B}$ by $h'(a) = b_a$. First of all, we must show that h' is in fact a homomorphism, which is quite analogous to the proof of the same fact for $\mathbb{A}(h)$. Given $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in A$,

$$\{h'(a) : a \in \sigma_\mathcal{A}(a_1, \ldots, a_n)\} = \{b_a : a \in \sigma_\mathcal{A}(a_1, \ldots, a_n)\} =$$

$$\sup\{\{b_a\} : a \in \sigma_\mathcal{A}(a_1, \ldots, a_n)\} = \sup\{h(\{a\}) : a \in \sigma_\mathcal{A}(a_1, \ldots, a_n)\} =$$

$$h(\sup\{\{a\} : a \in \sigma_\mathcal{A}(a_1, \ldots, a_n)\}),$$

given that h is continuous. Since it is an "almost homomorphism", this equals

$$\begin{aligned}
h(\sigma_\mathcal{A}(a_1, \ldots, a_n)) &= h(\sigma_{\mathbb{P}(\mathcal{A})}(\{a_1\}, \ldots, \{a_n\})) \\
&\subseteq \sigma_{\mathbb{P}(\mathcal{B})}(h(\{a_1\}), \ldots, h(\{a_n\})) \\
&= \sigma_{\mathbb{P}(\mathcal{B})}(\{b_{a_1}\}, \ldots, \{b_{a_n}\}) \\
&= \sigma_\mathcal{B}(b_{a_1}, \ldots, b_{a_n}) \\
&= \sigma_\mathcal{B}(h'(a_1), \ldots, h'(a_n)).
\end{aligned}$$

Now, when we consider $\mathbb{P}(h')$, we see that $\mathbb{P}(h')(\{a\}) = \{b_a\} = h(\{a\})$ for every atom $\{a\}$ of $\mathbb{P}(\mathcal{A})$, and so the restrictions of h and $\mathbb{P}(h')$ to atoms are the same, and therefore $\mathbb{A}(h) = \mathbb{A}(\mathbb{P}(h'))$. Since $\mathbb{A}$ is faithful, we discover that $h = \mathbb{P}(h')$ and, therefore, $\mathbb{P}$ is full. $\qquad\square$

Now it remains to be shown that $\mathbb{A}$ is also full. Given ord-algebras $\mathcal{A}$ and $\mathcal{B}$ over Σ, and a homomorphism $h : \mathbb{A}(\mathcal{A}) \to \mathbb{A}(\mathcal{B})$, we then define $h' : \mathcal{A} \to \mathcal{B}$ by

$$h'(a) = \sup\{h(c) : c \in A_a\}.$$

First of all, we must prove that h' is a homomorphism of ord-algebras, for which we shall need a few lemmas.

LEMMA 4.4. *In a complete, atomic and bottomless Boolean algebra $\mathcal{A}$, take a non-empty family of indexes I and, for every $i \in I$, $X_i \subseteq A$. Suppose we have $x_i = \sup X_i$, for $i \in I$, and $X = \bigcup\{X_i : i \in I\}$. Then, $\sup\{x_i : i \in I\} = \sup X$.*

PROOF: We define $a = \sup\{x_i : i \in I\}$ and $b = \sup X$: first, we show that a is an upper bound for X, so that $a \geq_{\mathcal{A}} b$. For every $x \in X$, we have that, since $X = \bigcup\{X_i : i \in I\}$, there exists $j \in I$ such that $x \in X_j$, and therefore $x_j \geq_{\mathcal{A}} x$. Since $a = \sup\{x_i : i \in I\}$, we have that $a \geq_{\mathcal{A}} x_j$, and by transitivity $a \geq_{\mathcal{A}} x$, and therefore a is indeed an upper bound for X.

Now we show that b is an upper bound for $\{x_i : i \in I\}$, and so $b \geq_{\mathcal{A}} a$ (and $a = b$). For every $i \in I$, we have that b is an upper bound for X_i, since $X_i \subseteq X$ and b is an upper bound for X, and therefore $b \geq_{\mathcal{A}} x_i$, since x_i is the smallest upper bound for X_i. It follows that b is indeed an upper bound for $\{x_i : i \in I\}$, what completes the proof. $\square$

LEMMA 4.5. *In a complete, atomic and bottomless Boolean algebra $\mathcal{A}$, for a non-empty $C \subseteq A$ one has that $\bigcup\{A_c : c \in C\} = A_{\sup C}$.*

PROOF: If $d \in A_c$ for a $c \in C$, d is an atom such that $d \leq_{\mathcal{A}} c$. Since $c \leq_{\mathcal{A}} \sup C$, $d \leq_{\mathcal{A}} \sup C$, and therefore d belongs to $A_{\sup C}$. Thus $\bigcup\{A_c : c \in C\} \subseteq A_{\sup C}$.

Conversely, suppose that $d \in A_{\sup C}$. Then, d is an atom such that $d \leq_{\mathcal{A}} \sup C$, and therefore $\inf\{d, \sup C\} = d$. It follows that the subset $C^d \subseteq C$, of $c \in C$ such that $\inf\{d, c\}$ exists, is not empty, by Proposition 1.7. But if $c \in C^d$, $\inf\{d, c\}$ exists, and since d is an atom, we have that $d \in A_c \subseteq \bigcup\{A_c : c \in C\}$, and from that $\bigcup\{A_c : c \in C\} = A_{\sup C}$. $\square$

Since $\sigma_{\mathcal{A}}(a_1, \ldots, a_n)$ is equal to the supremum of $\{\sigma_{\mathcal{A}}(c_1, \ldots, c_n) : (c_1, \ldots, c_n) \in A_{a_1} \times \cdots \times A_{a_n}\}$, from Lemma 4.5 we have that $A_{\sigma_{\mathcal{A}}(a_1, \ldots, a_n)}$ is equal to

$$\bigcup\{A_{\sigma_{\mathcal{A}}(c_1, \ldots, c_n)} : c_1 \in A_{a_1}, \ldots, c_n \in A_{a_n}\},$$

that is, we have the following lemma.

LEMMA 4.6. *For $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in A$,*

$$A_{\sigma_{\mathcal{A}}(a_1, \ldots, a_n)} = \bigcup\{A_{\sigma_{\mathcal{A}}(c_1, \ldots, c_n)} : c_i \in A_{a_i}\}.$$

THEOREM 4.7. $\mathbb{A}$ *is full.*

PROOF: First of all, we prove that h' is a homomorphism of ord-algebras.

(i) First, it is clear that h' maps atoms into atoms: if a is an atom, $A_a = \{a\}$ and

$$h'(a) = \sup\{h(c) : c \in A_a\} = \sup\{h(a)\} = h(a),$$

which is an atom since h is a map between $\mathbb{A}(\mathcal{A})$ and $\mathbb{A}(\mathcal{B})$.

(ii) h' is continuous: for any non-empty set $C \subseteq A$, we remember that $h'(c) = \sup\{h(d) : d \in A_c\}$, and from Lemmas 4.4 and 4.5 we get that

$$\begin{aligned}
\sup\{h'(c) : c \in C\} &= \sup\{\sup\{h(d) : d \in A_c\} : c \in C\} \\
&= \sup\bigcup\{\{h(d) : d \in A_c\} : c \in C\} \\
&= \sup\{h(d) : d \in A_{\sup C}\} \\
&= h'(\sup C).
\end{aligned}$$

(iii) Since $\{h(a) : a \in \sigma_{\mathbb{A}(\mathcal{A})}(a_1, \ldots, a_n)\} \subseteq \sigma_{\mathbb{A}(\mathcal{B})}(h(a_1), \ldots, h(a_n))$, given that h is a homomorphism of multialgebras, it follows from Lemma 4.6 that

$$\begin{aligned}
h'(\sigma_{\mathcal{A}}(a_1, \ldots, a_n)) &= \sup\{h(c) : c \in \bigcup\{A_{\sigma_{\mathcal{A}}(c_1,\ldots,c_n)} : c_i \in A_{a_i}\}\} \\
&= \sup\bigcup\{\{h(c) : c \in \sigma_{\mathbb{A}(\mathcal{A})}(c_1, \ldots, c_n)\} : c_i \in A_{a_i}\} \\
&\leq_{\mathcal{B}} \sup\bigcup\{\sigma_{\mathbb{A}(\mathcal{B})}(h(c_1), \ldots, h(c_n)) : c_i \in A_{a_i}\},
\end{aligned}$$

where we have used that, for atoms $c_1, \ldots, c_n$ of $\mathcal{A}$, $\sigma_{\mathbb{A}(\mathcal{A})}(c_1, \ldots, c_n) = A_{\sigma_{\mathcal{A}}(c_1,\ldots,c_n)}$. Since, for atoms $d_1, \ldots, d_n$ of $\mathcal{B}$, we also have that $\sigma_{\mathbb{A}(\mathcal{B})}(d_1, \ldots, d_n) = A_{\sigma_{\mathcal{B}}(d_1,\ldots,d_n)}$, this is equal to

$$\sup\bigcup\{A_{\sigma_{\mathcal{B}}(h(c_1),\ldots,h(c_n))} : c_i \in A_{a_i}\} = \sup\bigcup\{A_{\sigma_{\mathcal{B}}(h'(c_1),\ldots,h'(c_n))} : c_i \in A_{a_i}\}.$$

Since h' is continuous, $c_i \leq_{\mathcal{A}} a_i$, for every $i \in \{1, \ldots, n\}$, implies $h'(c_i) \leq_{\mathcal{B}} h'(a_i)$, and therefore $\sigma_{\mathcal{B}}(h'(c_1), \ldots, h'(c_n)) \leq_{\mathcal{B}} \sigma_{\mathcal{B}}(h'(a_1), \ldots, h'(a_n))$ for $(c_1, \ldots, c_n) \in A_{a_1} \times \cdots \times A_{a_n}$. It follows that the union, for $(c_1, \ldots, c_n)$ in $A_{a_1} \times \cdots \times A_{a_n}$, of $A_{\sigma_{\mathcal{B}}(h'(c_1),\ldots,h'(c_n))}$, is contained on $A_{\sigma_{\mathcal{B}}(h'(a_1),\ldots,h'(a_n))}$, and therefore

$$\begin{aligned}
\sup\bigcup\{A_{\sigma_{\mathcal{B}}(h'(c_1),\ldots,h'(c_n))} : c_i \in A_{a_i}\} &\leq_{\mathcal{B}} \sup A_{\sigma_{\mathcal{B}}(h'(a_1),\ldots,h'(a_n))} \\
&= \sigma_{\mathcal{B}}(h'(a_1), \ldots, h'(a_n)).
\end{aligned}$$

Now, for every atom a of $\mathcal{A}$, we have that $h'(a) = h(a)$, and therefore the restriction of h' to atoms coincides with h, that is, $\mathbb{A}(h') = h$, and since h was taken arbitrarily, $\mathbb{A}$ is full. $\square$

4.2. $\mathbb{P}$ and $\mathbb{A}$ are adjoint

It remains to be shown that $\mathbb{P}$ and $\mathbb{A}$ are adjoint. To this end, consider the bijections

$$\Phi_{\mathcal{B},\mathcal{A}} : Hom_{\mathbf{MAlg}(\Sigma)}(\mathbb{A}(\mathcal{B}), \mathcal{A}) \to Hom_{\mathbf{OAlg}(\Sigma)}(\mathcal{B}, \mathbb{P}(\mathcal{A})),$$

for $\mathcal{A}$ a Σ-multialgebra and $\mathcal{B}$ an ord-algebra over Σ, given by, for $h : \mathbb{A}(\mathcal{B}) \to \mathcal{A}$ a homomorphism and b an element of $\mathcal{B}$,

$$\Phi_{\mathcal{B},\mathcal{A}}(h)(b) = \{h(c) : c \in A_b\}.$$

LEMMA 4.8. $\Phi_{\mathcal{B},\mathcal{A}}(h)$ *is a homomorphism of ord-algebras.*

PROOF: (i) If b is an atom, $A_b = \{b\}$, and therefore $\Phi_{\mathcal{B},\mathcal{A}}(h)(b) = \{h(c) : c \in A_b\} = \{h(b)\}$, which is a singleton and therefore an atom of $\mathbb{P}(\mathcal{A})$.

(ii) Let D be a non-empty subset of $\mathcal{B}$. We have that

$$
\begin{aligned}
\Phi_{\mathcal{B},\mathcal{A}}(h)(\sup D) &= \{h(c) : c \in A_{\sup D}\} \\
&= \{h(c) : c \in \bigcup\{A_d : d \in D\}\} \\
&= \bigcup\{\{h(c) : c \in A_d\} : d \in D\} \\
&= \bigcup\{\Phi_{\mathcal{B},\mathcal{A}}(h)(d) : d \in D\} \\
&= \sup\{\Phi_{\mathcal{B},\mathcal{A}}(h)(d) : d \in D\},
\end{aligned}
$$

since $A_{\sup D} = \bigcup_{d \in D} A_d$ and the supremum in $\mathbb{P}(\mathcal{A})$ is simply the union.

(iii) For $\sigma \in \Sigma_n$ and $b_1, \ldots, b_n$ elements of $\mathcal{B}$, we have that

$$
\begin{aligned}
\Phi_{\mathcal{B},\mathcal{A}}(h)(\sigma_{\mathcal{B}}(b_1, \ldots, b_n)) &= \{h(c) : c \in A_{\sigma_{\mathcal{B}}(b_1, \ldots, b_n)}\} \\
&= \{h(c) : c \in \bigcup\{A_{\sigma_{\mathcal{B}}(c_1, \ldots, c_n)} : c_i \in A_{b_i}\}\} \\
&= \bigcup\{\{h(c) : c \in A_{\sigma_{\mathcal{B}}(c_1, \ldots, c_n)}\} : c_i \in A_{b_i}\},
\end{aligned}
$$

and, since $c_1, \ldots, c_n$ are atoms, this is equal to

$$\bigcup \{\{h(c) : c \in \sigma_{\mathbb{A}(\mathcal{B})}(c_1, \ldots, c_n)\} : c_i \in A_{b_i}\}$$
$$\subseteq \bigcup \{\sigma_{\mathcal{A}}(h(c_1), \ldots, h(c_n)) : c_i \in A_{b_i}\}$$
$$= \bigcup \{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) : a_i \in \{h(c) : c \in A_{b_i}\}\}$$
$$= \bigcup \{\sigma_{\mathcal{A}}(a_1, \ldots, a_n) : a_i \in \Phi_{\mathcal{B},\mathcal{A}}(h)(b_i)\}$$
$$= \sigma_{\mathbb{P}(\mathcal{A})}(\Phi_{\mathcal{B},\mathcal{A}}(h)(b_1), \ldots, \Phi_{\mathcal{B},\mathcal{A}}(h)(b_n)). \qquad \square$$

LEMMA 4.9. *The function $\Phi_{\mathcal{B},\mathcal{A}}$ is a bijection between $Hom_{\textbf{MAlg}(\Sigma)}(\mathbb{A}(\mathcal{B}), \mathcal{A})$ and $Hom_{\textbf{OAlg}(\Sigma)}(\mathcal{B}, \mathbb{P}(\mathcal{A}))$.*

PROOF: The functions $\Phi_{\mathcal{B},\mathcal{A}}$ are certainly injective: if $\Phi_{\mathcal{B},\mathcal{A}}(h) = \Phi_{\mathcal{B},\mathcal{A}}(h')$, for every atom b we have that

$$\{h(b)\} = \Phi_{\mathcal{B},\mathcal{A}}(h)(b) = \Phi_{\mathcal{B},\mathcal{A}}(h')(b) = \{h'(b)\},$$

and therefore $h = h'$.

For the surjectivity, take a homomorphism of ord-algebras $h : \mathcal{B} \to \mathbb{P}(\mathcal{A})$. We then define $h' : \mathbb{A}(\mathcal{B}) \to \mathcal{A}$ by $h'(b) = a$ for an atom b in $\mathcal{B}$, where $h(b) = \{a\}$. It is well-defined since a homomorphism of ord-algebras takes atoms to atoms, and the atoms of $\mathbb{P}(\mathcal{A})$ are exactly the singletons.

We must show that h' is indeed a homomorphism. For $\sigma \in \Sigma_n$ and atoms $b_1, \ldots, b_n$ in $\mathbb{A}(\mathcal{B})$ such that $h(b_i) = \{a_i\}$ for every $i \in \{1, \ldots, n\}$, we have that $h(\sigma_{\mathcal{B}}(b_1, \ldots, b_n)) \subseteq \sigma_{\mathbb{P}(\mathcal{A})}(h(b_1), \ldots, h(b_n))$, since h is a homomorphism of ord-algebras, and therefore

$$\{h'(b) : b \in \sigma_{\mathbb{A}(\mathcal{B})}(b_1, \ldots, b_n)\} = \{h'(b) : b \in A_{\sigma_{\mathcal{B}}(b_1, \ldots, b_n)}\}$$
$$= \bigcup \{h(b) : b \in A_{\sigma_{\mathcal{B}}(b_1, \ldots, b_n)}\}$$
$$= h(\sup A_{\sigma_{\mathcal{B}}(b_1, \ldots, b_n)})$$
$$= h(\sigma_{\mathcal{B}}(b_1, \ldots, b_n))$$
$$\subseteq \sigma_{\mathbb{P}(\mathcal{A})}(h(b_1), \ldots, h(b_n))$$
$$= \sigma_{\mathbb{P}(\mathcal{A})}(\{a_1\}, \ldots, \{a_n\})$$
$$= \sigma_{\mathcal{A}}(a_1, \ldots, a_n)$$
$$= \sigma_{\mathcal{A}}(h'(b_1), \ldots, h'(b_n)).$$

Finally, we state that $\Phi_{\mathcal{B},\mathcal{A}}(h') = h$ since, for any element b in $\mathcal{B}$, we have that

$$\Phi_{\mathcal{B},\mathcal{A}}(h')(b) = \{h'(c) : c \in A_b\} = \bigcup\{h(c) : c \in A_b\} = h(\sup A_b) = h(b),$$

and therefore each $\Phi_{\mathcal{B},\mathcal{A}}$ is, indeed, bijective. $\qquad\square$

THEOREM 4.10. $\mathbb{P}$ and $\mathbb{A}$ are adjoint.

PROOF: Given $\mathcal{A}$ and $\mathcal{C}$ two Σ-multialgebras, $\mathcal{B}$ and $\mathcal{D}$ two ord-algebras over Σ, $h : \mathcal{A} \to \mathcal{C}$ a homomorphism and $h' : \mathcal{D} \to \mathcal{B}$ a homomorphism of ord-algebras, we must now only prove that the following diagram commutes.

$$
\begin{array}{ccc}
Hom_{\mathbf{MAlg}(\Sigma)}(\mathbb{A}(\mathcal{B}), \mathcal{A}) & \xrightarrow{\ \Phi_{\mathcal{B},\mathcal{A}}\ } & Hom_{\mathbf{OAlg}(\Sigma)}(\mathcal{B}, \mathbb{P}(\mathcal{A})) \\
\downarrow{\scriptstyle Hom(\mathbb{A}(h'),h)} & & \downarrow{\scriptstyle Hom(h',\mathbb{P}(h))} \\
Hom_{\mathbf{MAlg}(\Sigma)}(\mathbb{A}(\mathcal{D}), \mathcal{C}) & \xrightarrow{\ \Phi_{\mathcal{D},\mathcal{C}}\ } & Hom_{\mathbf{OAlg}(\Sigma)}(\mathcal{D}, \mathbb{P}(\mathcal{C}))
\end{array}
$$

So, we take a homomorphism $g : \mathbb{A}(\mathcal{B}) \to \mathcal{A}$ and an element d of $\mathcal{D}$. We have that

$$Hom(h', \mathbb{P}(h))(\Phi_{\mathcal{B},\mathcal{A}}(g)) = \mathbb{P}(h) \circ \Phi_{\mathcal{B},\mathcal{A}}(g) \circ h',$$

and therefore the right side of the diagram gives us

$$\mathbb{P}(h) \circ \Phi_{\mathcal{B},\mathcal{A}}(g) \circ h'(d) = \mathbb{P}(h)(\{g(b) : b \in A_{h'(d)}\}) = \{h \circ g(b) : b \in A_{h'(d)}\}.$$

The left side gives us

$$\Phi_{\mathcal{D},\mathcal{C}}(h \circ g \circ \mathbb{A}(h'))(d) = \{h \circ g \circ \mathbb{A}(h')(e) : e \in A_d\} = \{h \circ g \circ h'(e) : e \in A_d\}.$$

If d is an atom, the right side becomes the singleton containing only $h \circ g \circ h'(d)$, since in this case $A_d = \{d\}$ and, given that h' preserves atoms, $A_{h'(d)} = \{h'(d)\}$. The left side becomes also the singleton formed by $h \circ g \circ h'(d)$, because again $A_d = \{d\}$. As a homomorphism of ord-algebras is determined by its action on atoms, we find that the left and right sides of the diagram are equal, and therefore the diagram commutes. As observed before, this proves that $\mathbb{A}$ and $\mathbb{P}$ are adjoint. $\qquad\square$

COROLLARY 4.11. The categories **MAlg**(Σ) and **OAlg**(Σ) are equivalent.

PROOF: Follows from the fact that $\mathbb{P}$ and $\mathbb{A}$ are an equivalence between the two categories, proven in Theorem 4.10. □

5.   Some consequences and related results

The result that **MAlg**(Σ) and **OAlg**(Σ) are equivalent has a few consequences, and related results, we would like to stress. First of all, we start by taking the empty signature: in that case, given that all multialgebras are non-empty, **MAlg**(Σ) becomes the category of non-empty sets **Set***, with functions between them as morphisms.

Meanwhile, **OAlg**(Σ) becomes the category with complete, atomic and bottomless Boolean algebras as objects (given we simply drop the operations from an ord-algebra over Σ), with continuous, atoms-preserving functions between them as morphisms. Notice this is very closely related to the equivalence between **CABA** and **Set**^{op}: the morphisms on the former are merely continuous functions, so the only extra requirement to the morphisms we are making is that they should preserve atoms. This, of course, allows one to exchange the opposite category of **Set** by **Set** itself (or rather **Set***).

A generalization of our result is to partial multialgebras. That is, pairs $\mathcal{A} = (A, \{\sigma_{\mathcal{A}}\}_{\sigma \in \Sigma})$ such that, if $\sigma \in \Sigma_n$, $\sigma_{\mathcal{A}}$ is a function from A^n to $\mathcal{P}(A)$ (no longer $\mathcal{P}(A) \setminus \{\emptyset\}$). In other words, a partial multialgebra is a multialgebra where operations may return the empty-set. Given partial Σ-multialgebras $\mathcal{A}$ and $\mathcal{B}$, a homomorphism between them is a function $h : A \to B$ such that, as is the case for homomorphisms for multialgebras,

$$\{h(a) : a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)\} \subseteq \sigma_{\mathcal{B}}(h(a_1), \ldots, h(a_n)),$$

for $\sigma \in \Sigma_n$ and $a_1, \ldots, a_n \in A$. The class of all partial Σ-multialgebras, with these homomorphisms between them as morphisms, becomes a category, which we shall denote by **PMAlg**(Σ).

It is easy to find an equivalence, much alike the one between **MAlg**(Σ) and **OAlg**(Σ), between **PMAlg**(Σ) and a category related to **OAlg**(Σ): it is sufficient to replace the requirement that, in an ord-algebra, $(A, \leq_{\mathcal{A}})$ is a complete, atomic and bottomless Boolean algebra by the requisite that it is actually a complete, atomic Boolean algebra, and accordingly, change

the morphisms in the correspondent category by requiring they preserve the supremum of any sets, not necessarily non-empty.

Finally, let us slightly modify the notion of homomorphism between Σ-multialgebras: a multihomomorphism h from $\mathcal{A}$ to $\mathcal{B}$, what may be written as $h : \mathcal{A} \rightarrow \mathcal{B}$ for simplicity, is a function $h : A \rightarrow \mathcal{P}(B) \setminus \{\emptyset\}$ that satisfies

$$\bigcup_{a \in \sigma_{\mathcal{A}}(a_1, \ldots, a_n)} h(a) \subseteq \bigcup_{(b_1, \ldots, b_n) \in h(a_1) \times \cdots \times h(a_n)} \sigma_{\mathcal{B}}(b_1, \ldots, b_n).$$

The category with: Σ-multialgebras as objects; multihomomorphisms as morphisms; and the composition of morphisms $h : \mathcal{A} \rightarrow \mathcal{B}$ and $g : \mathcal{B} \rightarrow \mathcal{C}$ given by, for an element a of $\mathcal{A}$, $g \circ h(a) = \bigcup_{b \in h(a)} g(b)$, will be denoted by **MMAlg**(Σ). If, in the category **OAlg**(Σ), we change morphisms by not longer demanding that they map atoms into atoms, it is easy to adapt the proof given in Section 4 to show that the resulting category is equivalent to **MMAlg**(Σ).

Conclusion and future work

As we explained before, the main results here presented involve an equivalence similar to the one between the categories of complete, atomic Boolean algebras and **Set**^{op}: the first is modified by addition of operations to said Boolean algebras, that are required to furthermore be compatible with the algebra's order; meanwhile, to the second we attach non-deterministic (yet still non-partial) operations, leading us to a category of multialgebras.

Although not specially complicated, this result is useful as it allows to treat non-deterministic matrices (Nmatrices) as, not precisely algebraic semantics, but mixed methods that combine both an algebraic component and one relative to its order. This may seem to increase the complexity of decision methods, but this sacrifice is made precisely to avoid non-determinism; and it is made, not because we distrust the use of multialgebras as semantics for non-classical logics, but as an alternative to those logicians that have philosophical objections against that very use.

More pragmatically, we feel encouraged to develop a further study of the categories of multialgebras, now from the viewpoint of categories of partially ordered sets, far better understood than the former ones. Moreover, we can now recast several non-deterministic characterizations of logics found in the literature in the terms here presented. Specifically, there are

several paraconsistent logics uncharacterizable by finite matrices, but characterized by finite Nmatrices, which can now have semantics presented only in classical terms of algebras and orders.

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