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SUP-HESITANT FUZZY INTERIOR IDEALS IN Γ-SEMIGROUPS

Abstract

In this paper, we defined the concept of $\mathcal{SUP}$-hesitant fuzzy interior ideals in Γ-semigroups, which is generalized of hesitant fuzzy interior ideals in Γ-semigroups. Additionally, we study fundamental properties of $\mathcal{SUP}$-hesitant fuzzy interior ideals in Γ-semigroups. Finally, we investigate characterized properties of those.

Keywords: SUP-hesitant fuzzy interior ideal, hesitant fuzzy interior ideal, interval valued fuzzy interior ideal.

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1. Introduction

The theory of fuzzy sets (FSs), considered by Zadeh in [27] has applications in mathematics, engineering, medical science, and other fields. Torra and Narukawa [25] extended the knowledge of a fuzzy set go to a hesitant fuzzy set (HFS) which is a function from a reference set to a power set of the unit interval and a generalization of intuitionistic fuzzy sets (IFSs) and interval-valued fuzzy sets (IvFSs) [26]. Then in 2015, Jun et al. [14] introduced the concept of HFSs and studied many algebraic structures, such as properties of hesitant fuzzy left (right, generalized bi-, bi-, two-sided) ideals of semigroups. In 1981, Sen introduced the concept of Γ-semigroup as a

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generalization of the plain semigroup and ternary semigroup. The many classical notions and results of (ternary) semigroups have been extended and generalized to Γ-semigroups, by many mathematicians. For instance, Dutta, and Davvaz [7, 8] studied the theory of Γ-semigroups via fuzzy subsets. Siripitukdet and Iampan [22, 23], Siripitukdet and Julatha [24], Dutta and Adhikari [8], Saha and Sen [20, 21], Hila, [10, 11] and Chinram [4, 5], and Uckun et al. [18] studied the theory of Γ-semigroup via intuitionistic fuzzy subsets. Abbasi et al. [1] introduced hesitant fuzzy left (resp., right, bi-, interior, and two-sided) Γ-ideals of Γ-semigroups. Julatha and Iampan [13] introduced a sup-hesitant fuzzy Γ-ideal, which is a general concept of an interval valued fuzzy Γ-ideal and a hesitant fuzzy Γ-ideal, of a Γ-semigroup and studied its properties via level sets, fuzzy sets, interval-valued fuzzy sets, and hesitant fuzzy sets. In 2018, Mosrijai et al., [16] presented the concept from HFSs in UP-algebras, namely $\mathcal{SUP}$-hesitant fuzzy UP-subalgebras (UP-filters, UP-ideals, strong UPideals). In 2019, Muhiuddin and Jun [17] introduced and studied the properties of $\mathcal{SUP}$-hesitant fuzzy subalgebras and their translations and extensions. In 2020, Muhiuddin et al. [17] studied the concept of $\mathcal{SUP}$-hesitant fuzzy ideals in BCK/BCI-algebras. In the same year, Harizavi and Jun [9] introduced $\mathcal{SUP}$-hesitant fuzzy quasi-associative ideal in BCI algebras. Later, Dey et al. [6] developed the concept of hesitant multi-fuzzy sets by combining the hesitant fuzzy set with the multi-fuzzy set. In 2021, Jittburus and Julatha [12] discussed the properties of $\mathcal{SUP}$-hesitant fuzzy ideals of semigroups and studied the characterizations in terms of sets, FSs, HFSs, and IvFSs. In 2022, P. Julatha and A. Iampan [13] studied the $\mathcal{SUP}$-hesitant fuzzy ideal in Γ-semigroup and considered the basic properties of those.

In this paper, we study the definition and properties of $\mathcal{SUP}$-hesitant fuzzy interior ideals in Γ-semigroups and investigate the properties of those.

2. Preliminaries

Throughout this paper, we denote a Γ-semigroup by $\mathcal{S}$.

In this section, we give some fundamental concepts about Γ-semigroups, fuzzy sets, intuitionistic fuzzy sets, interval valued fuzzy sets and hesitant fuzzy sets are presented. These notions will be helpful in later sections.

Let $\mathcal{S}$ and Γ be non-empty sets. Then $\mathcal{S}$ is called a Γ-*semigroup* $\mathcal{S}$ if there exists a function $\mathcal{S} \times \Gamma \times \mathcal{S} \to \mathcal{S}$ written as $(e_1, \alpha, e_2) \mapsto e_1\alpha e_2$

satisfying the axiom $(e_1\alpha e_2)\beta e_3 = e_1\alpha(e_2\beta e_3)$ for all $e_1, e_2, e_3 \in \mathcal{S}$ and $\alpha, \beta \in \Gamma$. A non-empty subset L of $\mathcal{S}$ is called a *subsemigroup* of $\mathcal{S}$ if $L\Gamma L \subseteq L$. A non-empty subset L of $\mathcal{S}$ is called a *left* (right) ideal of $\mathcal{S}$ if $\mathcal{S}\Gamma L \subseteq L$ ($L\Gamma\mathcal{S} \subseteq L$). By an Γ-*ideal* L of $\mathcal{S}$, we mean a left ideal and a right ideal of $\mathcal{S}$. A subsemigroup L of $\mathcal{S}$ is called a *interior ideal* of $\mathcal{S}$ if $\mathcal{S}\Gamma L\Gamma\mathcal{S} \subseteq L$.

A *fuzzy set* (FS) of a non-empty set $\mathcal{T}$ is a function $\omega : \mathcal{T} \to [0, 1]$.

Definition 2.1 ([15]). A FS ω of $\mathcal{S}$ is said to be a *fuzzy subsemigroup* (FSG) of $\mathcal{S}$ if $\omega(e_1\gamma e_2) \geq \omega(e_1) \wedge \omega(e_2)$ for all $e_1, e_2 \in \mathcal{S}$ and $\gamma \in \Gamma$.

Definition 2.2 ([19]). A FS ω of $\mathcal{S}$ is said to be a *fuzzy left (right) ideal* (FLI(FRI)) of $\mathcal{S}$ if $\omega(e_1\gamma e_2) \geq \omega(e_2)$ ($\omega(e_1\gamma e_2) \geq \omega(e_1)$) for all $e_1, e_2 \in \mathcal{S}$ and $\gamma \in \Gamma$. A FS ω of $\mathcal{S}$ is called an *fuzzy ideal* of $\mathcal{S}$ if it is both a fuzzy left ideal and a fuzzy right ideal of $\mathcal{S}$.

Definition 2.3 ([19]). A FS ω of $\mathcal{S}$ is said to be an *fuzzy interior ideal* (FII) of $\mathcal{S}$ if ω is a FSG and $\omega(e_1\gamma e_2\alpha e_3) \geq \omega(e_2)$ for all $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$.

An *intuitionistic fuzzy set* (IFS) $\mathcal{A}$ in $\mathcal{T}$ is the form $\mathcal{A} = \{e, \omega_{\mathcal{A}}, \vartheta_{\mathcal{A}} \mid e \in \mathcal{A}\}$ where $\omega_{\mathcal{A}} : \mathcal{T} \to [0, 1]$ and $\vartheta_{\mathcal{A}} : \mathcal{T} \to [0, 1]$ and where $0 \leq \omega_{\mathcal{A}}(e) + \vartheta_{\mathcal{A}}(e) \leq 1$ for all $e \in \mathcal{A}$ [2].

Definition 2.4 ([18]). An IFS $\mathcal{A} = (\omega_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ in $\mathcal{T}$ is called an intuitionistic fuzzy subemigroup (IFSG) of $\mathcal{S}$ if $\omega_{\mathcal{A}}(e_1\gamma e_2) \geq \max\{\omega_{\mathcal{A}}(e_1), \omega_{\mathcal{A}}(e_2)\}$ and $\vartheta_{\mathcal{A}}(e_1\gamma e_2) \leq \min\{\vartheta_{\mathcal{A}}(e_1), \vartheta_{\mathcal{A}}(e_2)\}$ for all $e_1, e_2 \in \mathcal{S}$ and $\gamma \in \Gamma$.

Definition 2.5 ([18]). An IFS $\mathcal{A} = (\omega_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ in $\mathcal{T}$ is called an intuitionistic fuzzy ideal (IFI) of $\mathcal{S}$ if $\omega_{\mathcal{A}}(e_1\gamma e_2) \leq \max\{\omega_{\mathcal{A}}(e_1), \omega_{\mathcal{A}}(e_2)\}$ and $\vartheta_{\mathcal{A}}(e_1\gamma e_2) \geq \min\{\vartheta_{\mathcal{A}}(e_1), \vartheta_{\mathcal{A}}(e_2)\}$ for all $e_1, e_2 \in \mathcal{S}$ and $\gamma \in \Gamma$.

Definition 2.6 ([18]). An IFS $\mathcal{A} = (\omega_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ in $\mathcal{T}$ is called an intuitionistic interior ideal (IFII) of $\mathcal{S}$ if $\mathcal{A} = (\omega_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ is an IFSG and $\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) \geq \omega_{\mathcal{A}}(e_2)$ and $\vartheta_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) \leq \vartheta_{\mathcal{A}}(e_2)$ for all $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$.

Let $\mathcal{C}[0, 1]$ be the set of all closed subintervals of $[0, 1]$, i.e.,

$$\mathcal{C}[0, 1] = \{\tilde{p} = [p^-, p^+] \mid 0 \leq p^- \leq p^+ \leq 1\}.$$

Let $\hat{p} = [p^-, p^+]$ and $\hat{q} = [q^-, q^+] \in \Omega[0, 1]$. Define the operations $\preceq$, $=$, $\curlywedge$ and $\curlyvee$ as follows:

(1) $\hat{p} \preceq \hat{q}$ if and only if $p^- \leq q^-$ and $p^+ \leq q^+$.

(2) $\hat{p} = \hat{q}$ if and only if $p^- = q^-$ and $p^+ = q^+$.

(3) $\hat{p} \curlywedge \hat{q} = [(p^- \wedge q^-), (p^+ \wedge q^+)]$.

(4) $\hat{p} \curlyvee \hat{q} = [(p^- \vee q^-), (p^+ \vee q^+)]$.

If $\hat{p} \succeq \hat{q}$, we mean $\hat{q} \preceq \hat{p}$.

Definition 2.7 ([19]). Let $\mathcal{T}$ be a non-empty set. Then the function $\hat{\omega} : \mathcal{T} \to \mathcal{C}[0,1]$ is called *interval valued fuzzy set* (shortly, IvFS) of $\mathcal{T}$.

Next, we shall give definitions of various types of interval valued fuzzy subsemigroups.

Definition 2.8 ([3]). An IvFS $\hat{\omega}$ of $\mathcal{S}$ is said to be an *interval valued fuzzy subsemigroup* (IvF subsemigroup) of $\mathcal{S}$ if $\hat{\omega}(e_1\gamma e_2) \succeq \hat{\omega}(e_1) \curlywedge \hat{\omega}(e_2)$ for all $e_1, e_2 \in \mathcal{S}$ and $\gamma \in \Gamma$.

Definition 2.9 ([3]). An IvFS $\hat{\omega}$ of $\mathcal{S}$ is said to be an *interval valued fuzzy left (right) ideal* (IvF left (right) ideal) of $\mathcal{S}$ if $\hat{\omega}(e_1\gamma e_2) \succeq \hat{\omega}(e_2)$ ($\hat{\omega}(e_1\gamma e_2) \succeq \hat{\omega}(e_1)$) for all $e_1, e_2 \in \mathcal{S}$ and $\gamma \in \Gamma$. An IvFS $\hat{\omega}$ of $\mathcal{S}$ is called an *IvF ideal* of $\mathcal{S}$ if it is both an IvF left ideal and an IvF right ideal of $\mathcal{S}$.

Definition 2.10 ([3]). An IvFS $\hat{\omega}$ of $\mathcal{S}$ is said to be an *interval valued fuzzy interior ideal* (IvF interior ideal) of $\mathcal{S}$ if $\hat{\omega}$ is an IvF subsemigroup and $\hat{\omega}(e_1\gamma e_2\alpha e_3) \succeq \hat{\omega}(e_2)$ for all $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$.

Let L be a non-empty subset of $\mathcal{T}$. *An interval valued characteristic function* ($\hat{\lambda}_L$) of L is defined by

$$\hat{\lambda}_L : \mathcal{T} \to \mathcal{C}[0,1], e \mapsto \begin{cases} \overline{1} & \text{if } eu \in L, \\ \overline{0} & \text{otherwise,} \end{cases}$$

for all $e \in \mathcal{T}$.

For two IvFSs $\hat{\omega}$ and $\hat{\vartheta}$ of $\mathcal{S}$, define the product $\hat{\omega} \circ \hat{\vartheta}$ as follows: for all $e \in \mathcal{S}$,

$$(\hat{\omega} \circ \hat{\vartheta})(e) = \begin{cases} \curlyvee_{e=tz} \{\hat{\omega}(t) \curlywedge \hat{\vartheta}(z)\} & \text{if } e = tz, \\ \\ \overline{0} & \text{otherwise.} \end{cases}$$

DEFINITION 2.11 ([14]). A *hesitant fuzzy set* (HFS) on a non-emptyset $\mathcal{T}$ is a function $\mathfrak{h} : \mathcal{T} \to \mathcal{P}([0, 1])$.

DEFINITION 2.12 ([1]). A HFS $\mathfrak{h}$ on $\mathcal{S}$ is called a *hesitant fuzzy subsemigroup* (HFSG) on $\mathcal{S}$ if it satisfies:

$$\mathfrak{h}(e_1\gamma e_2) \supseteq \mathfrak{h}(e_1) \cap \mathfrak{h}(e_2) \text{ for all } e_1, e_2 \in \mathcal{S} \text{ and } \gamma \in \Gamma.$$

DEFINITION 2.13 ([1]). A HFS $\mathfrak{h}$ on $\mathcal{S}$ is called a *hesitant fuzzy left (resp., right) ideal* on $\mathcal{S}$ if it satisfies:

$$\mathfrak{h}(e_1\gamma e_2) \supseteq \mathfrak{h}(x)(\mathfrak{h}(e_1) \supseteq \mathfrak{h}(e_2)) \text{ for all } e_1, e_2 \in \mathcal{S} \text{ and } \gamma \in \Gamma.$$

An HFS $\mathfrak{h}$ of $\mathcal{S}$ is called an *hesitant fuzzy ideal* of $\mathcal{S}$ if it is both a hesitant fuzzy left ideal and a hesitant fuzzy right ideal of $\mathcal{S}$.

DEFINITION 2.14 ([1]). A HFS $\mathfrak{h}$ on $\mathcal{S}$ is called a *hesitant fuzzy interior ideal* (HFII) on $\mathcal{S}$ if it satisfies:

$$\mathfrak{h} \text{ is a HFs and } \mathfrak{h}(e_1\gamma e_2\alpha e_3) \supseteq \mathfrak{h}(e_2) \text{ for all } e_1, e_2, e_3 \in \mathcal{S} \text{ and } \gamma, \alpha \in \Gamma.$$

Let L be a non-empty subset of $\mathcal{T}$. The *characteristic hesitant fuzzy set* (CH_L) of L is defined by

$$CH_L : \mathcal{T} \to \mathcal{P}([0, 1]), x \mapsto \begin{cases} [0,\ 1] & \text{if } e \in L, \\ \emptyset & \text{otherwise,} \end{cases}$$

for all $e \in \mathcal{T}$.

For two HFSs $\mathfrak{h}$ and $\mathfrak{g}$ of $\mathcal{S}$, define the product $\mathfrak{h} \circ \mathfrak{g}$ as follows: for all $e \in \mathcal{S}$,

$$(\mathfrak{h} \circ \mathfrak{g})(e) = \begin{cases} \bigcup\limits_{e=tz} \{\mathfrak{h}(t) \cap \mathfrak{g}(z)\} & \text{if } e = tz, \\ \emptyset & \text{otherwise.} \end{cases}$$

3. $\mathcal{SUP}$-hesitant fuzzy interior ideals in Γ-Semigroups

In this section, we define the concepts of $\mathcal{SUP}$-hesitant fuzzy interior ideals of $\mathcal{S}$ and characterize $\mathcal{SUP}$-hesitant fuzzy interior ideals of $\mathcal{S}$.

For any HFS $\mathfrak{h}$ on $\mathcal{T}$ and $\Theta \in \mathcal{P}[0, 1]$, define $\mathcal{SUP}\Theta$ and $\mathcal{S}[\mathfrak{h}; \Theta]$ by

$$\mathcal{SUP}\Theta = \begin{cases} \sup\Theta & \text{if } \Theta \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{S}[\mathfrak{h};\Theta] = \{x \in \mathcal{X} \mid \mathcal{SUP}\ \mathfrak{h}(x) \geq \mathcal{SUP}\Theta\}.$$

Then the following assertions are true:

(1) For every IvFS $\tilde{A}$ on $\mathcal{X}, \mathcal{SUP}\tilde{A}(x) = \sup\tilde{A}(x) = A^{+}(x), \forall x \in \mathcal{X}$.

(2) If $\Theta, \Upsilon \in \mathcal{P}[0,1]$ with $\Theta \subseteq \Upsilon$, then $\mathcal{SUP}\Theta \subseteq \mathcal{SUP}\Psi$ and $\mathcal{S}[\mathfrak{h};\Upsilon] \subseteq \mathcal{S}[\mathfrak{h};\Theta]$.

Definition 3.1. An HFS $\mathfrak{h}$ on $\mathcal{S}$ is called a SUP-hesitant fuzzy interior ideal of $\mathcal{S}$ related to Θ (Θ-$\mathcal{SUP}$-HFI) if the set $\mathcal{S}[\mathfrak{h};\Theta]$ is an interior ideal of $\mathcal{S}$. We call that $\mathfrak{h}$ is a $\mathcal{SUP}$-hesitant fuzzy interior ideal ($\mathcal{SUP}$-HFII) of $\mathcal{S}$ if $\mathfrak{h}$ is a Θ-$\mathcal{SUP}$-HFII of $\mathcal{S}, \forall\Theta \in \mathcal{P}[0,1]$ with $\mathcal{S}[\mathfrak{h};\Theta] \neq \emptyset$.

The following Lemmas are tools to prove Theorem 3.7.

Lemma 3.2. *If $\Theta, \Psi \in \mathcal{P}[0,1]$ with $\mathcal{SUP}\Theta = \mathcal{SUP}\Upsilon$ and $\mathfrak{h}$ is a Θ-$\mathcal{SUP}$-HFI of $\mathcal{S}$, then $\mathfrak{h}$ is a Ψ-$\mathcal{SUP}$-HFI of $\mathcal{S}$.*

Proof: Assume that $\Theta, \Upsilon \in \mathcal{P}[0,1]$ with $\mathcal{SUP}\Theta = \mathcal{SUP}\Upsilon$ and $\mathfrak{h}$ is a Θ-$\mathcal{SUP}$-HFI of $\mathcal{S}$. Then $\mathcal{SUP}\Theta \subseteq \mathcal{SUP}\Psi$ and $\mathcal{S}[\mathfrak{h};\Upsilon] \subseteq \mathcal{S}[\mathfrak{h};\Theta]$. Thus, by Definition 3.1, $\mathfrak{h}$ is a Υ-$\mathcal{SUP}$-HFI of $\mathcal{S}$. □

Lemma 3.3. *Every IvF interior ideal of $\mathcal{S}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

Proof: Assume that $\tilde{A}$ is an IvF interior ideal of $\mathcal{S}$ and let $\Theta \in \mathcal{P}[0,1]$ with $\mathcal{S}[\tilde{A};\Theta] \neq \emptyset$. Let $e_1, e_3 \in \mathcal{S}$, $e_2 \in \mathcal{S}[\tilde{A};\Theta]$ and $\gamma, \alpha \in \Gamma$. Then $\sup\tilde{A}(e_2) \geq \mathcal{SUP}\Theta$. Since $\tilde{A}$ is an IvF interior ideal of $\mathcal{S}$, we have $\mathcal{SUP}\Theta \leq \sup\tilde{A}(e_2) \preceq \tilde{A}(e_1\gamma e_2\alpha e_3)$. Thus, $e_1\gamma e_2\alpha e_3 \in \mathcal{S}[\tilde{A},\Theta]$. Hence, $\tilde{A}$ is an interior ideal of $\mathcal{S}$. So, $\tilde{A}$ is a Θ-$\mathcal{SUP}$-HFII of $\mathcal{S}$. Therefore, $\tilde{A}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

Lemma 3.4. *Every HFII of $\mathcal{S}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

Proof: Assume that $\mathfrak{h}$ is a HFII of $\mathcal{S}$ and let $\Theta \in \mathcal{P}[0,1]$ with $\mathcal{S}[\tilde{A};\Theta] \neq \emptyset$. Let $e_1, e_3 \in \mathcal{S}$ and $e_2 \in \mathcal{S}[\mathfrak{h};\Theta]$ and $\gamma, \alpha \in \Gamma$. Then $\mathfrak{h}(e_1\gamma e_2\alpha e_3) \supseteq \mathfrak{h}(e_2)$. Thus, $\mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) \geq \mathfrak{h}(e_2) \geq \mathcal{SUP}\Theta$ so $e_1\gamma e_2\alpha e_3 \in \mathcal{S}[\mathfrak{h};\Theta]$.

Hence, $\mathcal{S}[\mathfrak{h};\Theta]$ is an interior ideal of $\mathcal{S}$, and so $\mathfrak{h}$ is a Θ-$\mathcal{SUP}$-HFII of $\mathcal{S}$. Therefore, $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

THEOREM 3.5. *Let S be a regular Γ-semigroup $\mathcal{S}$. Then HFS $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ if and only if $\mathfrak{h}$ is a $\mathcal{SUP}$-HFI of $\mathcal{S}$.*

PROOF: It is a direct result from that a non-empty subset L of a regular Γ-semigroup $\mathcal{S}$ is an interior ideal of $\mathcal{S}$ if and only if L is an ideal of $\mathcal{S}$. □

For every HFS $\mathfrak{h}$ on $\mathcal{T}$ and $\Theta \in \mathcal{P}[0,1]$, we define the HFS $\mathcal{H}(\mathfrak{h};\Theta)$ on $\mathcal{T}$ by $\forall e \in \mathcal{T}$,

$$\mathcal{H}(\mathfrak{h};\Theta)(e) = \{r \in \Theta \mid \mathcal{SUP}\mathfrak{h}(e) \geq r\}.$$

We denote $\mathcal{H}(\mathfrak{h}; \bigcup_{e\in\mathcal{T}} \mathfrak{h}(e))$ by $\mathcal{H}_h$ and denote $\mathcal{H}(\mathfrak{h};[0,1])$ by $\mathcal{I}_h$. Then the following assertions are true: for all $e \in \mathcal{T}$;

(1) $\mathcal{I}_h$. is an IvFS on $\mathcal{S}$.

(2) $\mathfrak{h}(e) \subseteq \mathcal{H}_h \subseteq \mathcal{I}$.

(3) $\mathcal{SUP}\mathfrak{h}(e) = \mathcal{SUP}\mathcal{H}_h(x) = \mathcal{SUP}\mathcal{I}_h(e)$.

(4) $\mathcal{H}(\mathfrak{h},\Theta)(e) \subseteq \Theta$.

(5) $\mathcal{H}(\mathfrak{h},\Theta)(e) = \Theta$ if an only if $e \in \mathcal{S}[\mathfrak{h},\Theta]$.

LEMMA 3.6. *An HFS $\mathfrak{h}$ on $\mathcal{S}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ if and only if $\mathcal{H}(\mathfrak{h};\Theta)$ is an HFII of $\mathcal{S}, \forall\Theta \in \mathcal{P}[0,1]$.*

PROOF: Let $\Theta \in \mathcal{P}[0,1]$, $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$. Suppose that $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$, and let $r \in \mathcal{H}(\mathfrak{h};\Theta)(e_2)$. Then $a \in \mathcal{H}(\mathfrak{h};\Theta)(a)$. Thus, $\mathcal{SUP}(\mathfrak{h}(a)) \geq r \in \Theta$. Hence, $e_2 \in \mathcal{S}[\mathfrak{h}(e_2)]$. Since $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$, we have $e_1e_2e_3 \in \mathcal{S}[\mathfrak{h}(a)]$. Thus, $\mathcal{SUP}\mathfrak{h}(e_1e_2e_3) \geq \mathfrak{h}(e_1) \geq r \in \Theta$. Hence, $r \in \mathcal{H}(\mathfrak{h};\Theta)(e_1e_2e_3)$. Therefore, $\mathcal{H}(\mathfrak{h};\Theta)(e_2) \subseteq \mathcal{H}(\mathfrak{h};\Theta)(e_1e_2e_3)$. We conclude that $\mathcal{H}(\mathfrak{h};\Theta)$ is a HFII of $\mathcal{S}$.

In contrat, suppose that $\mathfrak{h}$ is a $\mathcal{H}(\mathfrak{h};\Theta)$ is a HFII of $\mathcal{S}$ and $e_2 \in \mathcal{S}[\mathfrak{h};\Theta]$, $e_1, e_3 \in \mathcal{S}$. Then $\mathcal{H}(\mathfrak{h},\Theta)(e_2) = \Theta$. Since $\mathfrak{h}$ is a $\mathcal{H}(\mathfrak{h};\Theta)$ is a HFII of $\mathcal{S}$ we have $\Theta = \mathcal{H}(\mathfrak{h},\Theta)(e_2) \subseteq \mathcal{H}(\mathfrak{h};\Theta)(e_2) \subseteq \mathcal{H}(\mathfrak{h};\Theta)(e_1e_2e_3)$ and so $\Theta \subseteq \mathcal{H}(\mathfrak{h};\Theta)(e_1e_2e_3)$. Hence $\mathcal{SUP}\mathfrak{h}(e_1e_2e_3) \geq \mathcal{SUP}\Theta$. Thus $e_1e_2e_3 \in \mathcal{S}[\mathfrak{h};\Theta]$.

Therefore $\mathcal{S}[\mathfrak{h};\Theta]$ is an interior ideal of $\mathcal{S}$. This implies that $\mathfrak{h}$ is a $\Theta-\mathcal{SUP}$-HFII of $\mathcal{S}$. Thus $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

The following theorem is a result of Lemma 3.3, 3.4, and 3.6.

THEOREM 3.7. *Let $\mathfrak{h}$ is a HFS in $\mathcal{K}$. Then the following statements are equivalent.*

(1) *$\mathcal{H}_h$ is an HFII of $\mathcal{S}$.*

(2) *$\mathcal{H}_h$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

(3) *$\mathcal{I}_h$ is a $\mathcal{SUP}$-HFII of $\mathcal{K}$.*

(4) *$\mathcal{I}_h$ is an HFII of $\mathcal{S}$.*

(5) *$\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

(6) *$\mathcal{I}_h$ is an IvFII of $\mathcal{S}$.*

PROOF: By Lemma 3.4, we get that, $1 \Rightarrow 2$ and $3 \Rightarrow 4$.

By Lemma 3.6, we get that, $5 \Rightarrow 2$ and $5 \Rightarrow 6$.

By Lemma 3.3, we get that, $3 \Rightarrow 6$.

Now, we proof $1 \Rightarrow 5$. Let $\Theta \in \mathcal{P}[0,1], e_1, e_2e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$. Then $\mathcal{SUP}\mathcal{H}_h(e_2) = \mathcal{SUP}\mathfrak{h}(e_2) \geq \mathcal{SUP}\Theta$. Thus, $e_2 \in \mathcal{S}[\mathcal{H}_h;\Theta]$. So, $\mathcal{S}[\mathcal{H}_h;\Theta]$ is an interior ideal of $\mathcal{S}$ with $e_1\gamma e_2\alpha e_3 \in \mathcal{S}[\mathcal{H}_h;\Theta]$ which implies that $\mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) = \mathcal{SUP}\mathcal{H}_h(e_1\gamma e_2\alpha e_3) \geq \mathcal{SUP}\Theta$. Hence, $e_1\gamma e_2\alpha e_3 \in \mathcal{S}[\mathfrak{h};\Theta]$. Therefore $\mathcal{S}[\mathfrak{h};\Theta]$ is an interior ideal of $\mathcal{S}$. We conclude that $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.

For $1 \Rightarrow 6$, let $e_1, e_2e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$. Then $e_2 \in S[\mathfrak{h};\mathfrak{h}(e)]$. Thus, $\mathcal{SUP}\mathfrak{h}(e_2) \leq \mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3)$. Hence, $\mathcal{I}_h(e_2) = [0, \mathcal{SUP}\mathfrak{h}(e_2)]$. So, $\mathcal{I}_h(e_2) \preceq \mathcal{I}_h(e_1\gamma e_2\alpha e_3)$. Therefore, $\mathcal{I}_h$ is an IvFBII of $\mathcal{S}$.

The proof of $2 \Rightarrow 6$ is similar to $1 \Rightarrow 5$. □

In [12], the author define $\mathcal{F}_\mathfrak{h}$ in $\mathcal{T}$ by $\mathcal{F}_\mathfrak{h} = \mathcal{SUP}\mathfrak{h}(x)$ for all $x \in \mathcal{T}$.

THEOREM 3.8. *An HFS $\mathfrak{h}$ on $\mathcal{K}$ is a $\mathcal{SUP}$-HFBII of $\mathcal{S}$ if and only if $\mathcal{F}_h$ is a FII of $\mathcal{S}$.*

PROOF: Let $e_1, e_2e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$. Then $\mathfrak{h}(e_2) = \Theta$ for some $\Theta \in \mathcal{P}[0,1]$. Thus, $e_2 \in \mathcal{S}[\mathfrak{h};\Theta]$. By assumption, we have $e_1\gamma e_2\alpha e_3 \in \mathcal{S}[\mathfrak{h};\Theta]$. Hence, $\mathcal{F}_h(e_1\gamma e_2\alpha e_3) = \mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) \geq \mathcal{SUP}\Theta = \mathcal{SUP}(\mathfrak{h}(e_2)) = \mathfrak{h}(a) = \mathcal{F}_h(e_2)$. Therefore, $\mathcal{F}_h$ is a fuzzy interior ideal of $\mathcal{S}$.

In contrat, let $\Theta \in \mathcal{P}[0,1], e_2 \in \mathcal{S}[\mathfrak{h};\Theta], e_1, e_3 \in \mathcal{S}$. Then $\mathcal{SUP}\mathfrak{h}(e_1e_2e_3) = \mathcal{F}_h(e_1\gamma e_2\alpha e_3) \geq \mathcal{F}_h(e_2) = \mathcal{SUP}h(e_2) \geq \mathcal{SUP}\Theta$. This implies that $e_1\gamma e_2\alpha e_3 \in \mathcal{S}[\mathfrak{h};\Theta]$. Hence, $\mathcal{S}[\mathfrak{h};\Theta]$ is an interior ideal of $\mathcal{S}$. So, $\mathfrak{h}$ is a Θ-$\mathcal{SUP}$-HFII of $\mathcal{S}$. Therefore, $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

The following result is an immediate consequence of Theorem 3.8.

Corollary 3.9. An HFS $\mathfrak{h}$ in $\mathcal{S}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ if and only if $\mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) = \mathfrak{h}(e_2)$ for all $e_1, e_2e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$.

For any IFS $\mathcal{A} = (\omega_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ on $\mathcal{T}$ and $\Theta \in \mathcal{P}[0,1]$, we define the HFS $\mathcal{H}^{\Theta}_{\mathcal{A}}$ on $\mathcal{T}$ and IvFS $\mathcal{I}_{\mathcal{A}}$ in $\mathcal{A}$

$$\mathcal{H}^{\Theta}_{\mathcal{A}}(e) = \left\{ t \in \Theta \mid \frac{\vartheta_{\mathcal{A}}(e)}{2} \leq t \leq \frac{1+\omega_{\mathcal{A}}(e)}{2} \right\}$$

and

$$\mathcal{I}_{\mathcal{A}}(e) = \left[\frac{1-\vartheta_{\mathcal{A}}(e)}{2}, \frac{1+\omega_{\mathcal{A}}(e)}{2} \right]$$

for all $e \in \mathcal{T}$.

Theorem 3.10. *Suppose that* $\mathcal{A} = (\omega_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ *be an IFS in* $\mathcal{S}$. *The following are equivalent.*

(1) $\mathcal{A}$ *is an IFII of* $\mathcal{S}$.

(2) $\mathcal{H}^{\Theta}_{\mathcal{A}}$ *is a HFII of* $\mathcal{S}$ *for all* $\Theta \in \mathcal{P}[0,1]$.

(3) $\mathcal{I}_{\mathcal{A}}$ *is an IvFII of* $\mathcal{K}$.

Proof: 1. ⇒ 2. Suppose that $\mathcal{A}$ is an IFII of $\mathcal{S}$ and $\Theta \in \mathcal{P}[0,1]$. Let $e_1, e_2, e_3 \in \mathcal{S}$, $\gamma, \alpha \in \Gamma$ and $t \in \mathcal{H}^{\Theta}_{\mathcal{A}}(e_2)$. Then $t \in \Theta$ and $\frac{\vartheta_{\mathcal{A}}}{2} \leq t \leq \frac{1+\omega_{\mathcal{A}}}{2}$. Since $\mathcal{A}$ is an IFII of $\mathcal{S}$, we have

$$\frac{\vartheta_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2} \leq \vartheta_{\mathcal{A}}(e_2) \leq t \leq \frac{1+\omega_{\mathcal{A}}(e_2)}{2} \leq \frac{1+\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2}.$$

Thus, $t \in \mathcal{H}^{\Theta}_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)$. Hence, $\mathcal{H}^{\Theta}_{\mathcal{A}}(e_2) \subseteq \mathcal{H}^{\Theta}_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)$.
Therefore, $\mathcal{H}^{\Theta}_{\mathcal{A}}$ is an HFII of $\mathcal{S}$.

2. ⇒ 1. Suppose that $\mathcal{H}^{\Theta}_{\mathcal{A}}$ is am HFII of $\mathcal{S}$, and $\mathcal{A}$ is not an IFII of $\mathcal{S}$. Then there are $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$ such that $\omega_{\mathcal{A}}(e_1e_2e_3) < \omega_{\mathcal{A}}(e_2)$. Choose $t = \frac{1}{4}(\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) + \omega_{\mathcal{A}}(e_2))$. We have $\frac{1}{2} + t \in [0,1]$

and $\frac{\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2} < t < \omega_{\mathcal{A}}(e_2)$. Thus, $\frac{\vartheta_{\mathcal{A}}(e_2)}{2} \leq \frac{1}{2} < \frac{1}{2} + t < \frac{1+\omega_{\mathcal{A}}(e_2)}{2}$. So, $\frac{1}{2} + t \in \mathcal{H}^{\Theta}_{\mathcal{A}}(e_2)$ Since $\mathcal{H}^{\Theta}_{\mathcal{A}}$ is an HFII of $\mathcal{S}$, we have $\mathcal{H}^{[0,1]}_{\mathcal{A}}$ is an HFII on $\mathcal{S}$. It implies that $\frac{1}{2} + t \in \mathcal{H}^{[0,1]}_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)$. Hence, $\frac{1}{2} + t \leq \frac{1+\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2}$ and

$$\begin{aligned}\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) &= 2\left(\frac{1+\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2}\right) - 1\\ &\geq 2\left(\frac{1}{2} + t\right)\\ &= 2t\\ &> \omega_{\mathcal{A}} e_1\gamma e_2\alpha e_3).\end{aligned}$$

It is a contradiction. Hence, $\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) \geq \omega_{\mathcal{A}}(e_2)$. Therefore, $\mathcal{A}$ is an IFII of $\mathcal{S}$.

1. $\Rightarrow$ 3. Suppose that $\mathcal{A}$ is an IFII of $\mathcal{S}$. Let $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$. Then $\frac{1-\vartheta_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2} \geq \frac{1-\vartheta_{\mathcal{A}}(e_2)}{2} = \frac{1-\vartheta_{\mathcal{A}}(e_2)}{2}$ and $\frac{1+\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2} \geq \frac{1+\omega_{\mathcal{A}}(e_2)}{2} = \frac{1+\omega_{\mathcal{A}}(e_2)}{2}$. Thus, $\mathcal{I}_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) \succeq \mathcal{I}_{\mathcal{A}}(e_2)$. Hence, $\mathcal{I}_{\mathcal{A}}$ is an IvFII of $\mathcal{S}$.

3. $\Rightarrow$ 1. Suppose that $\mathcal{I}_{\mathcal{A}}$ is an IvFII of $\mathcal{K}$, and let $e_1, e_2, e_3 \in \mathcal{S}$. Then $\mathcal{I}_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) \succeq \mathcal{I}_{\mathcal{A}}(e_2)$. Thus, $\frac{1-\vartheta_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2} \geq \frac{1-\vartheta_{\mathcal{A}}(e_2)}{2}$ and $\frac{1+\omega_{\mathcal{A}}(e_1\gamma e_2\alpha e_3)}{2} \geq \frac{1+\omega_{\mathcal{A}}(e_2)}{2}$. Hence, $\omega_{\mathcal{A}}(e_1e_2e_3) \geq \omega_{\mathcal{A}}(e_2)$ and $\vartheta_{\mathcal{A}}(e_1\gamma e_2\alpha e_3) \leq \vartheta_{\mathcal{A}}(e_2)$. Therefore, $\mathcal{A}$ is an IFII of $\mathcal{S}$. □

Corollary 3.11. Suppose that $\mathcal{A} = (\omega_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ be an IFS in $\mathcal{S}$. The followings are equivalent.

(1) $\mathcal{H}^{\Theta}_{\mathcal{A}}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ for all $\Theta \in \mathcal{P}[0,1]$.

(2) $\mathcal{I}_{\mathcal{A}}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.

For any HFS $\mathfrak{h}$ on $\mathcal{T}$, the HFS $\mathfrak{h}^*$ is defined by $\mathfrak{h}^*(e) = \{1 - \mathcal{SUP}\mathfrak{h}(e)\}$ for all $e \in \mathcal{T}$. We call $\mathfrak{h}^*$ a **supermum complement** [16] of $\mathfrak{h}$ on $\mathcal{T}$. Then $\mathcal{SUP}\mathfrak{h}^*(e) = 1 - \mathcal{SUP}\mathfrak{h}(e)$ for all $e \in \mathcal{T}$. Hence, $(\mathcal{F}_{\mathfrak{h}}, \mathcal{F}_{\mathfrak{h}^*})$ is an IFS in $\mathcal{T}$.

Theorem 3.12. *An HFS $\mathfrak{h}$ on $\mathcal{S}$ is a $\mathcal{SUP}-$HFII of $\mathcal{S}$ if and only if $(\mathcal{F}_h, \mathcal{F}^*_h)$ is an IFII of $\mathcal{S}$.*

Proof: Suppose that $\mathfrak{h}$ is a $\mathcal{SUP}-$HFII of $\mathcal{S}$, and let $e_1, e_2, e_3 \in \mathcal{S}$, $\gamma, \alpha \in \Gamma$. Then, by Theorem 3.8,

$$\mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) = \mathcal{F}_{\mathfrak{h}}(e_1\gamma e_2\alpha e_3) \geq \mathfrak{h}(e_2) = \mathcal{SUP}\mathfrak{h}(e_2).$$

and

$$\mathcal{F}_{\mathfrak{h}^*}(e_1\gamma e_2\alpha e_3) = 1 - \mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) \leq 1 - \mathcal{SUP}\mathfrak{h}(e_2) = \mathcal{F}_{\mathfrak{h}^*}(e_2).$$

Hence, $(\mathcal{F}_h, \mathcal{F}_h^*)$ is an IFII of $\mathcal{S}$.

Conversely, suppose that $(\mathcal{F}_h, \mathcal{F}_h^*)$ is an IFII of $\mathcal{S}$. Then $\mathcal{F}_h$ is FII of $\mathcal{S}$. Thus, by Theorem 3.8, $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

For HFS $\mathfrak{h}$ on $\mathcal{T}$ and $t \in [0,1]$, define

$$\mathcal{U}_{\mathcal{SUP}}(\mathfrak{h}; t) = \{e \in \mathcal{T} \mid \mathcal{SUP}\mathfrak{h}(e) \geq t\}$$

and

$$\mathcal{L}_{\mathcal{SUP}}(\mathfrak{h}; t) = \{e \in \mathcal{T} \mid \mathcal{SUP}\mathfrak{h}(e) \leq t\}.$$

We call the $\mathcal{U}_{\mathcal{SUP}}$ a *$\mathcal{SUP}$-upper t-level subset* and call the $\mathcal{L}_{\mathcal{SUP}}$ a *$\mathcal{SUP}$-lower t-level subset* [16] of $\mathfrak{h}$.

THEOREM 3.13. *Let $\mathfrak{h}$ is an HFS on $\mathcal{S}$. Then the following statements holds;*

(1) *$\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ if and only if $\mathcal{U}_{\mathcal{SUP}}(\mathfrak{h}; t)$ is either empty of an interior ideal of $\mathcal{S}$ for all $t \in [0,1]$.*

(2) *$\mathfrak{h}^*$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ if and only if $\mathcal{L}_{\mathcal{SUP}}(\mathfrak{h}; t)$ is either empty of an interior ideal of $\mathcal{S}$ for all $t \in [0,1]$.*

PROOF:

(1) Suppose that $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ and $t \in [0,1]$ such that $\mathcal{U}_{\mathcal{SUP}}(\mathfrak{h}; t) \neq \emptyset$. Choose $\Theta = \{t\}$. Then $\mathcal{S}[\mathfrak{h}, \Theta] = \mathcal{U}_{\mathcal{SUP}}(\mathfrak{h}; t) \neq \emptyset$. By assumption, we have $\mathcal{U}_{\mathcal{SUP}}(\mathfrak{h}; t) = \mathcal{S}[\mathfrak{h}, \Theta]$ is an interior ideal of $\mathcal{S}$.

   Conversely, suppose that $\mathcal{U}_{\mathcal{SUP}}(\mathfrak{h}; t)$ is either empty of an interior ideal of $\mathcal{S}$ for all $t \in [0,1]$ and $\Theta \in \mathcal{P}[0,1]$ such that $\mathcal{S}[\mathfrak{h}, \Theta] \neq \emptyset$. Choose $t = \mathcal{SUP}\Theta$. Then $\mathcal{U}_{\mathcal{SUP}}(\mathfrak{h}; t) = \mathcal{S}[\mathfrak{h}, \Theta] \neq \emptyset$. By assumption, we have $\mathcal{S}[\mathfrak{h}, \Theta] = \mathcal{U}_{\mathcal{SUP}}(\mathfrak{h}; t)$ is an interior ideal of $\mathcal{S}$. Thus, $\mathfrak{h}$ is a Θ-$\mathcal{SUP}$-HFII of $\mathcal{S}$. Hence, $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.

(2) Suppose that $\mathfrak{h}^*$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ and $t \in [0,1]$ such that $\mathcal{L}_{\mathcal{SUP}}(\mathfrak{h}; t) \neq \emptyset$. Choose $\Upsilon = \{1 - t\}$. Then $\mathcal{S}[\mathfrak{h}^*, \Upsilon] = \mathcal{L}_{\mathcal{SUP}}(\mathfrak{h}; t) \neq \emptyset$. By assumption, we have $\mathcal{L}_{\mathcal{SUP}}(\mathfrak{h}; t) = \mathcal{S}[\mathfrak{h}^*, \Upsilon]$ is an interior ideal of $\mathcal{S}$.

Conversely, suppose that $\mathcal{L}_{\mathcal{SUP}}(\mathfrak{h};t)$ is either empty of an interior ideal of $\mathcal{S}$ for all $t \in [0,1]$ and $\Upsilon \in \mathcal{P}[0,1]$ such that $\mathcal{S}[\mathfrak{h}^*, \Upsilon] \neq \emptyset$. Choose $t = 1 - \mathcal{SUP}\Upsilon$. Then

$$\mathcal{L}_{\mathcal{SUP}}(\mathfrak{h};t) = \mathcal{S}[\mathfrak{h}^*, \Upsilon] \neq \emptyset.$$

By assumption, we have $\mathcal{S}[\mathfrak{h}^*, \Upsilon] = \mathcal{L}_{\mathcal{SUP}}(\mathfrak{h};t)$ is an interior ideal of $\mathcal{S}$. Thus, $\mathfrak{h}^*$ is a Ψ-$\mathcal{SUP}$-HFII of $\mathcal{S}$. Hence, $\mathfrak{h}^*$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

For $\Theta, \Psi \in \mathcal{P}[0,1]$ with $\mathcal{SUP}\Theta < \mathcal{SUP}\Psi$, define a function $H_L^{(\Theta,\Upsilon)}$ as follows:

$$H_L^{(\Theta,\Psi)}\mathcal{T} \to \mathcal{P}([0,1]), e \mapsto \begin{cases} \Upsilon & \text{if } e \in I, \\ \Theta & \text{otherwise,} \end{cases}$$

Theorem 3.14. *Let L be a non-empty subset of $\mathcal{S}$ and $\Theta, \Upsilon \in \mathcal{P}[0,1]$ with $\mathcal{SUP}\Theta < \mathcal{SUP}$. Then L is an interior ideal of $\mathcal{S}$ if and only if $\mathcal{H}_L^{(\Theta,\Upsilon)}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

Proof: Suppose that L is an interior ideal of $\mathcal{S}$ and $\mathcal{SUPH}_L^{(\Theta,\Upsilon)}(e_1e_2e_3) < \mathcal{SUPH}_L^{(\Theta,\Upsilon)}(e_2)$ for some $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$. Then $\mathcal{SUPH}_L^{(\Theta,\Upsilon)}(e_2) = \mathcal{SUP}\Upsilon$, which implies that $e_2 \in L$. Since L is an interior ideal of $\mathcal{S}$, we have $e_1\gamma e_2\alpha e_3 \in L$, and so

$$\mathcal{SUPH}_L^{(\Theta,\Upsilon)}(e_1\gamma e_2\alpha e_3) = \mathcal{SUP}\Upsilon = \mathcal{SUPH}_L^{(\Theta,\Upsilon)}(e_2).$$

It is a contradiction. Hence, $\mathcal{SUPH}_L^{(\Theta,\Upsilon)}(e_1\gamma e_2\alpha e_3) \geq \mathcal{SUPH}_L^{(\Theta,\Upsilon)}(e_2)$, for all $e_1, e_2, e_3 \in \mathcal{S}$ and $\gamma, \alpha \in \Gamma$. By Theorem 3.8, $\mathcal{H}_L^{(\Theta,\Upsilon)}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.

Conversely, let $e_1, e_3 \in \mathcal{S}$, $e_2 \in L$ and $\gamma, \alpha \in \Gamma$. Then $\mathcal{H}_L^{(\Theta,\Upsilon)}(e_2) = \Upsilon$. Since $\mathcal{H}_L^{(\Theta,\Upsilon)}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$, by Theorem 3.9, we have $\mathcal{H}_L^{(\Theta,\Upsilon)}(e_1\gamma e_2\alpha e_3) \geq \mathcal{SUPH}_L^{(\Theta,\Upsilon)}(e_2) = \mathcal{SUP}\Upsilon$, which implies that $e_1\gamma e_2\alpha e_3 \in L$. Hence, L is an interior ideal of $\mathcal{S}$. □

Corollary 3.15. Let I be a non-empty subset of $\mathcal{K}$. Then, the following statements are equivalent.

(1) L is an interior ideal of $\mathcal{K}$.

(2) $\tilde{\lambda}_L$ is a $\mathcal{SUP}$-HFII of $\mathcal{K}$.

(3) CH_L is a $\mathcal{SUP}$-HFII of $\mathcal{K}$.

4. $\mathcal{SUP}$-hesitant fuzzy translations

In this section, we define of $\mathcal{SUP}$-hesitant fuzzy translations of $\mathcal{SUP}$-HFIIs of semigroups and discuss the cencepts of extensions and intensions of $\mathcal{SUP}$-HFIIs.

For an HFS $\mathfrak{h}$ on $\mathcal{T}$, let $\mathcal{K}_{\mathfrak{h}} := 1 - \sup\{\mathcal{SUP}\mathfrak{h}(e) \mid e \in \mathcal{T}\}$.

Let $t \in [0, \mathcal{K}_{\mathfrak{h}}]$, and we say that an HFS g on $\mathcal{T}$ is $\mathcal{SUP}$-hesitnat fuzzy t^{+}-traslation ($\mathcal{SUP}$-HFT$_t^+$) of $\mathfrak{h}$ if $\mathcal{SUP}\mathfrak{h}(e) + t$ for all $e \in \mathcal{T}$. Then $\mathfrak{h}$ is a $\mathcal{SUP}$-HFT$_0^+$ of $\mathfrak{h}$, and in the case that ρ_1 and ρ_2 are $\mathcal{SUP}$-HFT$_t^+$ of $\mathfrak{h}$, we see that $\mathcal{SUP}\rho_1(e) = \mathcal{SUP}\rho_2(e)$ for all $e \in \mathcal{T}$ but ρ_1 may be not equal to ρ_2.

THEOREM 4.1. *Let $\mathfrak{h}$ be a $\mathcal{SUP}$-HFII of $\mathcal{S}$ and $t \in [0, \mathcal{K}_{\mathfrak{h}}]$. Then every $\mathcal{SUP}$-HFT$_t^+$ of $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

PROOF: Suppose that ρ is a $\mathcal{SUP}$-HFT$_t^+$ of $\mathfrak{h}$, and let $e_1, e_2, e_3 \in \mathcal{S}$, $\gamma, \alpha \in \Gamma$. Then

$$\mathcal{SUP}\rho(e_1\gamma e_2\alpha e_3) = \mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) + t \geq \mathcal{SUP}\mathfrak{h}(e_2) + t = \mathcal{SUP}\mathfrak{h}(e_2).$$

Thus, by Corollary 3.9, ρ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

THEOREM 4.2. *Let $\mathfrak{h}$ be an HFII of $\mathcal{S}$ such that it is a $\mathcal{SUP}$-HFT$_t^+$ is $\mathcal{SUP}$-HFII of $\mathcal{S}$ for some $t \in [0, \mathcal{K}_{\mathfrak{h}}]$. Then $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

PROOF: Suppose that a $\mathcal{SUP}$-HFT$_t^+$ ρ of $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ when $t \in [0, \mathcal{K}_{\mathfrak{h}}]$. Then for all $e_1, e_2, e_3\mathcal{S}$ and $\gamma, \alpha \in \Gamma$,

$$\mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) = \mathcal{SUP}\rho(e_1\gamma e_2\alpha e_3) - t \geq \mathcal{SUP}\rho(e_2) - t = \mathcal{SUP}\rho(e_2).$$

Thus, by Corollary 3.9, $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

THEOREM 4.3. *Let $\mathfrak{h}$ be an HFS on $\mathcal{S}$ and $t \in [0, \mathcal{K}_{\mathfrak{h}}]$. Then a $\mathcal{SUP}$-HFT$_t^+$ of $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ if and only if $\mathcal{U}_{\mathcal{SUP}}(\mathfrak{h}; m - t)$ either empty or an interior ideal of $\mathcal{S}$ for all $m \in [t, 1]$.*

PROOF: (⇒) By Theorem 3.13. 1.

(⇐) Let ρ be a $\mathcal{SUP}$-HFT$_t^+$ of $\mathfrak{h}$ and $e_1, e_2, e_3 \in \mathcal{S}$, $\gamma, \alpha \in \Gamma$. Choose $m := \mathcal{SUP}\rho(e_2)$. Then $m - t = \mathcal{SUP}\rho(e_2) - t = \mathcal{SUP}\mathfrak{h}(e_2)$. Thus, $e_2 \in \mathcal{U}_{\mathcal{SUP}}(\mathfrak{h}; m - t)$. By assumption, $e_1\gamma e_2\alpha e_3 \in \mathcal{U}_{\mathcal{SUP}}(\mathfrak{h}; m - t)$. Hence,

$\mathcal{SUP}\rho(e_1\gamma e_2\alpha e_3) = \mathcal{SUP}\mathfrak{h}(e_1\gamma e_2\alpha e_3) + t \geq m = \mathcal{SUP}\rho(e_2)$. By Corollary 3.9, $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$. □

For an HFS $\mathfrak{h}$ on $\mathcal{S}$, define $\pm_{\mathfrak{h}} := \inf\{\mathcal{SUP}\mathfrak{h}(e) \mid e \in \mathcal{S}\}$.

For $t \in [0, \pm_{\mathfrak{h}}]$ an HFS g of $\mathcal{S}$ is said to be $\mathcal{SUP}$-hesitant fuzzy t^--translation ($\mathcal{SUP}$-HFT$_{t^-}$) of $\mathfrak{h}$ if $\mathcal{SUP}\rho(e) = \mathcal{SUP}\mathfrak{h}(e) - t$ for all $e \in \mathcal{S}$. Then $\mathfrak{h}$ is a $\mathcal{SUP}$-HFT$_{0^-}$ of $\mathfrak{h}$.

THEOREM 4.4. *Let $\mathfrak{h}$ be a $\mathcal{SUP}$-HFII of $\mathcal{S}$ and $t \in [0, \pm_{\mathfrak{h}}]$. Then every $\mathcal{SUP}$-HFT$_{t^-}$ of $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

PROOF: It follows Theorem 4.1. □

THEOREM 4.5. *Let $\mathfrak{h}$ be an HFS on $\mathcal{S}$ such that its $\mathcal{SUP}$-HFT$_{t^-}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$ for some $t \in [0, \pm_{\mathfrak{h}}]$. Then $\mathfrak{h}$ is a $\mathcal{SUP}$-HFII of $\mathcal{S}$.*

PROOF: It follows Theorem 4.2. □

5. Conclusion

In this paper, we study the results for $\mathcal{SUP}$-hesitant fuzzy interior ideals in Γ-semigroups. Finally, we get the relation of HFBII, $\mathcal{SUP}$-HFII and IvFII in Γ-semigroup in Theorem 3.7. In future work, we can study other results in these algebraic structures.

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