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Pedro del Valle-Inclan

HARMONY AND NORMALISATION IN BILATERAL LOGIC[1]

Abstract

In a recent paper del Valle-Inclan and Schlöder argue that bilateral calculi call for their own notion of proof-theoretic harmony, distinct from the usual (or 'unilateral') ones. They then put forward a specifically bilateral criterion of harmony, and present a harmonious bilateral calculus for classical logic.

In this paper, I show how del Valle-Inclan and Schlöder's criterion of harmony suggests a notion of normal form for bilateral systems, and prove normalisation for two (harmonious) bilateral calculi for classical logic, $\mathbf{HB}_1$ and $\mathbf{HB}_2$. The resulting normal derivations have the usual desirable features, like the separation and subformula properties. $\mathbf{HB}_1$-normal form turns out to be strictly stronger that the notion of normal form proposed by Nils Kürbis, and $\mathbf{HB}_2$-normal form is neither stronger nor weaker than a similar proposal by Marcello D'Agostino, Dov Gabbay, and Sanjay Modgyl.

Keywords: bilateralism, normalisation, harmony.

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1. Introduction

According to inferentialists the meaning of logical vocabulary is given by the rules governing its use in inferences. There is nothing more to the meaning of, for example, disjunction, than the rules governing when to infer, and what to infer from, certain sentences containing 'or'. It follows that one can define a connective by laying down rules that govern it, like the introduction and elimination rules of natural deduction systems.

Inferentialism faces an objection first posed by Arthur Prior [8]. Consider the binary operator 'tonk' defined by the following rules:

$$\frac{A}{A \text{ tonk } B}\,(\text{tonk I}) \qquad\qquad \frac{A \text{ tonk } B}{B}\,(\text{tonk E})$$

By chaining an application of (tonk I) with an application of (tonk E) one can deduce two arbitrary sentences from each other. According to Prior, inferentialists have to conclude that any sentence follows from any other. Inferentialists, on their part, typically reject the assumption that *any* set of rules adequately defines a connective. They hold that there is something wrong with the rules for 'tonk', something that makes it an illegitimate piece of vocabulary. This has given rise to the search for a criterion to determine which rules are acceptable definitions, a project which has come to be known, following Dummett [3], as the search for a criterion of *proof-theoretic harmony*.

The most common approach to harmony appeals to an intuitive notion of 'balance'. A set of introduction and elimination rules is balanced if the elimination rules are neither too strong nor too weak with respect to the introductions (and vice versa). The elimination rule for tonk, for instance, is held to be too strong with respect to its introduction rule. The idea is that (tonk E) allows one to derive 'too much' from 'A tonk B', given what (tonk I) requires in order to derive such a sentence. Over the years a host of non-equivalent explications of this intuitive notion of balance have been put forward (see Steinberger [10] for a brief overview). By and large they all have something in common: the usual formalisations of classical logic come out disharmonious. Thus, or so it is argued, inferentialism is incompatible with classical logic.

Ian Rumfitt [9] has argued that bilateralism can solve this incompatibility. According to Rumfitt's bilateralism the speech acts of assertion and rejection should be taken as primitive, rather than analysed in terms of each other. Furthermore, he argues, the meaning of classical connectives must be given bilaterally, by means of rules governing the assertion *and* the rejection of sentences containing them. By stipulating assertive and rejective rules for each connective, he is able to provide a calculus for classical logic that satisfies the usual requirements of harmony.

Rumfitt's position has been recently challenged by del Valle-Inclan and Schlöder [2]. They argue that bilateral calculi require their own notion of harmony, distinct from the standard (or 'unilateral') ones. Thus, although Rumfitt's system is harmonious *according to criteria fit for unilateral systems*, this is not enough to vindicate classical logic from an inferentialist point of view. They propose a bilateral criterion of harmony and show, using a result by Fernando Ferreira [4], that Rumfitt's system is not harmonious according to it. To solve the problem, they put forward a new Rumfitt-style formalisation of classical logic.

The aim of this paper is to explore the relation between del Valle-Inclan and Schlöder's criterion of harmony, on the one hand, and normalisation on the other. I will first show how their harmony criterion suggests a natural notion of normal form for bilateral calculi. Then, I will show that their calculus, as well as a closely related one, normalise. Derivations in normal form have the usual desirable features; the subformula and separation properties, in particular, can be obtained as corollaries of normalisation. Finally, I will briefly compare the present notion of normal form with proposals by Nils Kürbis [6] and Marcello D'Agostino, Dov Gabbay, and Sanjay Modgyl [1].

The paper is structured as follows: Section 2 recaps Rumfitt's position and del Valle-Inclan and Schlöder's criticism. Sections 3 and 4 prove normalisation and corollaries for two (harmonious) bilateral calculi for classical logic. Section 5 compares the present normalisation results with previous ones, and Section 6 concludes.

2. Rumfitt, bilateralism and harmony

There are two core tenets to Rumfitt's bilateralism. The first is that assertion and rejection are distinct, primitive speech acts that serve to express different attitudes towards propositional content (assent and dissent, respectively). The second is that both assertion and rejection play a role in our inferential practice. From an inferentialist point of view, it follows that to specify the meaning of a connective one must give rules that govern both the assertion and rejection of sentences containing it. Rumfitt does this by means of a natural deduction calculus for signed formulae, that is, standard formulae preceded by force indicators '+' and '−'. If A is a propositional formula, $+A$ is to be interpreted as the assertion of A, and $-A$ as its rejection; force indicators cannot be iterated or embedded. Rumfitt proposes the following operational rules for the classical connectives:[2]

Conjunction:

$$\frac{+A_1 \qquad +A_2}{+A_1 \wedge A_2}\,(+\wedge\mathrm{I}) \qquad\qquad \frac{-A_i}{-A_1 \wedge A_2}\,(-\wedge\mathrm{I})$$

$$\frac{+A_1 \wedge A_2}{+A_i}\,(+\wedge\mathrm{E}) \qquad\qquad \frac{-A_1 \wedge A_2 \qquad \begin{array}{c}[-A_1]^1\\ \mathcal{D}_1\\ \varphi\end{array} \qquad \begin{array}{c}[-A_2]^1\\ \mathcal{D}_2\\ \varphi\end{array}}{\varphi}\,(-\wedge\mathrm{E})^1$$

Disjunction:

$$\frac{+A_i}{+A_1 \vee A_2}\,(+\vee\mathrm{I}) \qquad\qquad \frac{-A_1 \qquad -A_2}{-A_1 \vee A_2}\,(-\vee\mathrm{I})$$

$$\frac{+A_1 \vee A_2 \qquad \begin{array}{c}[+A_1]^1\\ \mathcal{D}_1\\ \varphi\end{array} \qquad \begin{array}{c}[+A_2]^1\\ \mathcal{D}_2\\ \varphi\end{array}}{\varphi}\,(+\vee\mathrm{E})^1 \qquad\qquad \frac{-A_1 \vee A_2}{-A_i}\,(-\vee\mathrm{E})$$

Implication:

$$\frac{\begin{array}{c}[+A_1]^1\\ \mathcal{D}\\ +A_2\end{array}}{+A_1 \to A_2}\,(+\to\mathrm{I})^1 \qquad\qquad \frac{+A_1 \to A_2 \qquad +A_1}{A_2}\,(+\to\mathrm{E})$$

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[2] A note about notation: roman letters range over unsigned formulae, greek letters over signed formulae, brackets indicate discharged assumptions, and both vacuous and multiple discharges are allowed. When there are two formulae separated by '/' below the horizontal line, as in rule $(-\to\mathrm{E})$, an application of the rule in question can conclude *either* formulae, not both simultaneously (all rules are single-conclusion).

$$\frac{+A_1 \qquad -A_2}{-A_1 \to A_2}\,(-\to\mathrm{I}) \qquad\qquad \frac{-A_1 \to A_2}{+A_1/-A_2}\,(-\to\mathrm{E})$$

Negation:

$$\frac{-A}{+\neg A}\,(+\neg\mathrm{I}) \qquad\qquad \frac{+A}{-\neg A}\,(-\neg\mathrm{I})$$

$$\frac{+\neg A}{-A}\,(+\neg\mathrm{E}) \qquad\qquad \frac{-\neg A}{+A}\,(-\neg\mathrm{E})$$

In addition to operational rules Rumfitt's calculus contains *coordination principles*. These are rules that govern the interaction between '+' and '−', rather than specific connectives. They are meant to capture our conventions regarding the assertion and rejection of the same content. Rumfitt's coordination principles are (Rejection) and *Smilean reductio*:

$$\frac{+A \qquad -A}{\bot}\,(\text{Rejection}) \qquad \frac{\begin{matrix}[+A]^1\\ \mathcal{D}\\ \bot\end{matrix}}{-A}\,(\mathrm{SR}_1)^1 \qquad \frac{\begin{matrix}[-A]^1\\ \mathcal{D}\\ \bot\end{matrix}}{+A}\,(\mathrm{SR}_2)^1$$

The principle of (Rejection) states that the assertion and rejection of the same content are incompatible.[3] The two halves of *Smilean reductio* state (respectively) that if the assertion of a formula leads to absurdity one may reject it, and if the rejection of a formula leads to absurdity, then one may assert it. On top of this, *Smilean reductio* also encodes a form of explosion, through the vacuous discharge of assumptions. A more perspicuous representation of Rumfitt's commitments about the interplay between assertion and rejection can be given by the following coordination principles of *Explosion* and *Bilateral Excluded Middle*:

$$\frac{+A \qquad -A}{\varphi}\,(\text{ex}) \qquad \frac{\begin{matrix}[+A]^1\\ \mathcal{D}_1\\ \varphi\end{matrix} \qquad \begin{matrix}[-A]^1\\ \mathcal{D}_2\\ \varphi\end{matrix}}{\varphi}\,(\text{bem})^1$$

[3] Note that Rumfitt, following Tennant [11], takes '$\bot$' as a *punctuation sign* indicating a logical dead end. It is not a sentence, and therefore cannot be signed, embedded in formulae, or appear as a topmost node in derivations.

It is routine to check that:

Remark 2.1. (Rejection), (SR1) and (SR_2) are derivable from (ex) and (bem) and vice versa.

Rumfitt's operational rules satisfy all the standard criteria of harmony, and it is intuitively clear why this should be so. Take the rules for negation: in order to apply ($+\neg$ I) one needs to derive a sentence of the form $-A$, which is exactly what one gets from an application of ($+\neg$ E). Similarly, in order to apply ($-\neg$ I) one needs to derive a sentence of the form $+A$, which is precisely what an application of ($+\neg$ E) yields. In other words, its operational rules of the same sign are inverses of each other. Something similar, of course, applies to the other connectives.

Del Valle-Inclan and Schlöder [2] argue that Rumfitt-style bilateral calculi call for a more demanding notion of harmony. In unilateral natural deduction what one can do with a connective is determined by operational rules alone; the relation between operational rules, then, is all that unilateral harmony needs to take into account. In bilateral calculi, however, coordination principles permit further inferential steps. And crucially, this means that connectives whose operational rules are balanced according to all the usual standards can become tonk-like when they interact with coordination principles. They give the following connective 'bink' as an example:

$$\frac{+A \qquad -A}{+\,\mathrm{bink}\,A}\ (+\,\mathrm{bink}\ \mathrm{I}) \qquad \frac{+\,\mathrm{bink}\,A}{+A}\ (+\,\mathrm{bink}\ \mathrm{E}_1) \qquad \frac{+\,\mathrm{bink}\,A}{-A}\ (+\,\mathrm{bink}\ \mathrm{E}_2)$$

$$\frac{-A}{-\,\mathrm{bink}\,A}\ (-\,\mathrm{bink}\ \mathrm{I}) \qquad \frac{-\,\mathrm{bink}\,A}{-A}\ (-\,\mathrm{bink}\ \mathrm{E})$$

The assertive introduction and elimination rules of bink are inverses of each other, and so are its rejective rules. Indeed, bink is harmonious according to all the usual (unilateral) standards of harmony. If bink interacts with *Smilean reductio*, however, it trivialises the calculus it is part of. The following derivation, for instance, shows that there is a proof of $-A$ for any A:

$$\dfrac{\dfrac{\dfrac{\dfrac{[+\,\text{bink}\,A]^1}{+A}\,(+\,\text{bink E}_1) \qquad \dfrac{[+\,\text{bink}\,A]^1}{-A}\,(+\,\text{bink E}_2)}{\bot}}{-\,\text{bink}\,A}\,(\text{SR}_1)^1}{-A}\,(-\,\text{bink E})$$

Examples like this show that a bilateral criterion of harmony must take into consideration *both* the relation between introduction and elimination rules, on the one hand, and the relation between operational rules and coordination principles on the other. Del Valle-Inclan and Schlöder propose the following criterion of bilateral harmony:

> **Bilateral harmony:** A connective **c** is bilaterally harmonious iff (i) (+**c**I) and (+**c**E) are unilaterally harmonious; (ii) (−**c**I) and (−**c**E) are unilaterally harmonious; (iii) all coordination principles are preserved by the rules for **c** (i.e. when all coordination principles are restricted to atomic sentences, all their instances for sentences containing **c** as their main operator are derivable).

To put it simply: whatever unilateral harmony may be, bilateral harmony is *that* plus preservation of all coordination principles. For further examples, and a more thorough defence of the criterion, the reader is referred to [2].

Fernando Ferreira [4] has shown that Rumfitt's operational rules do not preserve *Smilean reductio*. Rumfitt's system, therefore, is not harmonious. To solve the problem del Valle Inclan and Schlöder propose slight modifications to the rejective rules for conjunction and the assertive rules for disjunction. Their rules for rejected conjunctions are:[4]

$$\dfrac{\begin{matrix}[+A]^1\\ \mathcal{D}\\ -B\end{matrix}}{-A\wedge B}\,(-\wedge\text{I})^1 \qquad \dfrac{-A\wedge B \qquad +A}{-B}\,(-\wedge\text{E})$$

[4]These rules for conjunction are also independently discussed in Nils Kürbis' [5]

Their rules for asserted disjunctions are:

$$\dfrac{\begin{array}{c}[-A]^1\\ \mathcal{D}\\ +B\end{array}}{+A\vee B}\,(+\vee\mathrm{I})^1 \qquad \dfrac{+A\vee B \qquad -A}{+B}\,(+\vee\mathrm{E})$$

These rules are analogous to the usual rules for the material conditional, and it is easy to check that they are harmonious according the usual unilateral criteria. In addition, they preserve all the coordination principles we have considered. More generally, call the set of Rumfitt's operational rules with the present modifications **B**. Then:

Remark 2.2. All the rules in **B** preserve *Smilean reductio*, (Rejection), (ex) and (bem).

Remark 2.2. follows trivially from the normalisation results to be proved below.

In what follows I will refer to the calculus consisting of **B**, (SR_1), (SR_2) and (Rejection) as $\mathbf{HB}_1$, and the calculus consisting of **B**, (ex) and (bem) as $\mathbf{HB}_2$. It follows from the observation that the modified operational rules are unilaterally harmonious, combined with Remark 2.2, that $\mathbf{HB}_1$ and $\mathbf{HB}_2$ are harmonious in del Valle-Inclan and Schlöder's sense. It is also routine to check that:

Remark 2.3. $\mathbf{HB}_1$ and $\mathbf{HB}_2$ are equivalent to Rumfitt's calculus (i.e. φ is derivable from Γ in Rumfitt's calculus iff it is derivable from Γ in $\mathbf{HB}_1$ and $\mathbf{HB}_2$).

$\mathbf{HB}_1$ and $\mathbf{HB}_2$, then, are bilaterally harmonious formalisations of classical logic. We can finally examine the relation between harmony and normalisation. I will discuss $\mathbf{HB}_1$ first, and deal with $\mathbf{HB}_2$ in Section 4.

3. Harmony and normalisation: $\mathbf{HB}_1$

All derivations in normal form, according to Dag Prawitz's original result [7], share a central feature: no formula occurrence in them is simultaneously the consequence of an introduction rule and the major premise of an elimination. This is usually thought to be related to harmony. The idea is that if the operational rules of a connective are 'balanced', one should gain nothing by first introducing and then immediately eliminating a connective. Therefore, it should be possible to eliminate all such steps within a derivation.[5] This is the core principle behind normalisation, and so for normal derivations in $\mathbf{HB}_1$ we also require that:

> **(i)** No conclusion of an I-rule is a major premise of an E-rule.

The notion of bilateral harmony proposed by del Valle-Inclan and Schlöder suggests a similar principle, this time regarding the interaction between operational rules and coordination principles. Their idea is that a connective is 'balanced' with respect to a coordination principle if one can lay down the coordination principle for atoms and prove it for complex sentences. To reflect this at the level of derivations we should, as before, require that applications of coordination principles to complex formulae should be eliminable. In other words, that for normal derivations in $\mathbf{HB}_1$:

> **(ii)** Coordination principles are applied only to atoms.

Clauses **(i)** and **(ii)** are enough to ensure the separation property for normal derivations. They are not, however, enough to secure the stronger subformula property, as the following derivation shows:

$$\dfrac{\dfrac{\dfrac{+p \qquad -p}{\bot}}{+q}\,(\mathrm{SR}_2)^0 \qquad \dfrac{\dfrac{+p \qquad -p}{\bot}}{-q}\,(\mathrm{SR}_1)^0}{\dfrac{\bot}{+r}\,(\mathrm{SR}_2)^0}$$

The derivation satisfies **(i)** and **(ii)**, but contains signed formula occurrences $+q$ and $-q$ which are subformulae of neither the assumptions nor the conclusion. This is due to the fact that the form of explosion encoded by *Smilean reductio* is used twice, consecutively. To avoid this kind of configuration in normal derivations, and thus ensure the subformula property, we need only require that:

> **(iii)** No conclusion of *Smilean reductio* is a premise of (Rejection).

[5]This is not the whole story. In the presence of, for example, the usual rules for disjunction, one can introduce a connective and eliminate it a few steps below in the derivation, rather than immediately after the introduction. Because of the modified operational rules of $\mathbf{HB}_1$, however, this cannot happen in normal derivations, so we need not worry about it.

Putting everything together, we have the following definition.

Definition 3.1. (Normal form)
A derivation in $\mathbf{HB}_1$ is in normal form if in it: **(i)** No conclusion of an I-rule is a major premise of an E-rule. **(ii)** Coordination principles are applied only to atoms. **(iii)** No conclusion of *Smilean reductio* is a premise of (Rejection).

Formula occurrences that infringe clauses **(i)**, **(ii)** and **(iii)** are called **maximal operational formulae**, **maximal coordination formulae** and **ancillary maximal formulae**, respectively.

The rest of this section proves normalisation and corollaries for $\mathbf{HB}_1$. The first step is providing appropriate reduction procedures. Since the rules for disjunction and conjunction are analogous, I will not explicitly provide reduction steps involving the latter. It may be useful to keep the overall normalisation strategy in mind when examining the reduction steps. Reductions for maximal coordination formulae may create maximal operational formulae of the same complexity. Reduction steps for maximal operational formulae, on the other hand, may create new ancillary maximal formulae only. Finally, reduction steps for *atomic* ancillary maximal formulae create no new maximal formulae of any kind. Therefore, the normalisation process reduces maximal coordination formulae first, followed by maximal operational formulae, and then ancillary maximal formulae.

3.1. Operational reductions

Negation:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ -A\end{matrix}}{+\neg A}\,(+\neg\ \mathrm{I})}{-A}\,(+\neg\ \mathrm{E}) \quad\rightsquigarrow\quad \begin{matrix}\mathcal{D}_1\\ -A\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ +A\end{matrix}}{-\neg A}\,(-\neg\ \mathrm{I})}{+A}\,(-\neg\ \mathrm{E}) \quad\rightsquigarrow\quad \begin{matrix}\mathcal{D}_1\\ +A\end{matrix}$$

Implication:

$$\dfrac{\dfrac{\begin{matrix}[+A]^1\\ \mathcal{D}_1\\ +B\end{matrix}}{+A\to B}\,(+\to \mathrm{I})^1 \qquad \begin{matrix}\mathcal{D}_2\\ +A\end{matrix}}{+B}\,(+\to \mathrm{E}) \quad \rightsquigarrow \quad \begin{matrix}\mathcal{D}_2\\ +A\\ \mathcal{D}_1\\ +B\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ +A\end{matrix}\qquad\begin{matrix}\mathcal{D}_2\\ -B\end{matrix}}{-A\to B}\,(-\to\mathrm{I})}{+A/-B}\,(-\to\mathrm{E}) \quad\rightsquigarrow\quad \begin{matrix}\mathcal{D}_{1/2}\\ +A/-B\end{matrix}$$

Disjunction:

$$\dfrac{\dfrac{\begin{matrix}[-A]^1\\ \mathcal{D}_1\\ +B\end{matrix}}{+A\vee B}\,(+\vee\ \mathrm{I})^1 \qquad \begin{matrix}\mathcal{D}_2\\ -A\end{matrix}}{+B}\,(+\vee\ \mathrm{E}) \quad \rightsquigarrow \quad \begin{matrix}\mathcal{D}_2\\ -A\\ \mathcal{D}_1\\ +B\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ -A\end{matrix}\qquad\begin{matrix}\mathcal{D}_2\\ -B\end{matrix}}{-A\vee B}\,(-\vee\ \mathrm{I})}{-A/-B}\,(-\vee\ \mathrm{E}) \quad\rightsquigarrow\quad \begin{matrix}\mathcal{D}_{1/2}\\ -A/-B\end{matrix}$$

3.2. Reducing (rejection) to atomic applications

Negation:

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ +\neg A\end{matrix}\qquad \begin{matrix}\mathcal{D}_2\\ -\neg A\end{matrix}}{\bot}\,(\mathrm{Rej}) \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ +\neg A\end{matrix}}{-A}\qquad \dfrac{\begin{matrix}\mathcal{D}_2\\ -\neg A\end{matrix}}{+A}}{\bot}\,(\mathrm{Rej})$$

Implication:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ +A\to B\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ -A\to B\end{array}}{\perp}\,(\mathrm{Rej}) \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ +A\to B\end{array}\qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ -A\to B\end{array}}{+A}}{+B}\qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ -A\to B\end{array}}{-B}}{\perp}\,(\mathrm{Rej})$$

Disjunction:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ +A\vee B\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ -A\vee B\end{array}}{\perp}\,(\mathrm{Rej}) \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ +A\vee B\end{array}\qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ -A\vee B\end{array}}{-A}}{+B}\qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ -A\vee B\end{array}}{-B}}{\perp}\,(\mathrm{Rej})$$

3.3. Reducing *Smilean reductio* to atomic applications

Negation: (the other case is analogous)

$$\dfrac{\begin{array}{c}[+\neg A]^1\\ \mathcal{D}\\ \perp\end{array}}{-\neg A}\,(\mathrm{SR}_1)^1 \qquad\rightsquigarrow\qquad \dfrac{\dfrac{\begin{array}{c}\dfrac{[-A]^1}{+\neg A}\\ \mathcal{D}\\ \perp\end{array}}{+A}\,(\mathrm{SR}_2)^1}{-\neg A}$$

Implication:

$$\dfrac{\begin{array}{c}[+A\to B]^1\\ \mathcal{D}\\ \perp\end{array}}{-A\to B}\,(\mathrm{SR}_1)^1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[+A]^1 \qquad [-A]^2}{\perp}}{+B}\,(\mathrm{SR}_2)^0}{+A\to B}\,(+\to\mathrm{I})^1\\ \mathcal{D}\\ \perp\end{array}}{+A}\,(\mathrm{SR}_2)^2 \qquad \dfrac{\begin{array}{c}\dfrac{[+B]^3}{+A\to B}\\ \mathcal{D}\\ \perp\end{array}}{-B}\,(\mathrm{SR}_1)^3}{-A\to B}$$

$$\dfrac{\begin{array}{c}[-A\to B]^1\\ \mathcal{D}\\ \bot\end{array}}{+A\to B}\,(\mathrm{SR}_2)^1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{[+A]^1\qquad[-B]^2}{-A\to B}\\ \mathcal{D}\\ \bot\end{array}}{-A}\,(\mathrm{SR}_1)^1 \qquad [+A]^3}{\bot}\,(\mathrm{SR}_2)^2}{+B}}{+A\to B}\,(+\to\mathrm{I})^3$$

Disjunction:

$$\dfrac{\begin{array}{c}[+A\lor B]^1\\ \mathcal{D}\\ \bot\end{array}}{-A\lor B}\,(\mathrm{SR}_1)^1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[-A]^1\qquad[+A]^2}{\bot}}{+B}\,(\mathrm{SR}_2)^0}{+A\lor B}\,(+\lor\mathrm{I})^1\\ \mathcal{D}\\ \bot\end{array}}{-A}\,(\mathrm{SR}_2)^2 \qquad \dfrac{\begin{array}{c}\dfrac{[+B]^3}{+A\lor B}\\ \mathcal{D}\\ \bot\end{array}}{-B}\,(\mathrm{SR}_1)^3}{-A\lor B}$$

$$\dfrac{\begin{array}{c}[-A\lor B]^1\\ \mathcal{D}\\ \bot\end{array}}{+A\lor B}\,(\mathrm{SR}_2)^1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\dfrac{[-A]^2\qquad[-B]^1}{-A\lor B}\\ \mathcal{D}\\ \bot\end{array}}{+B}\,(\mathrm{SR}_1)^1}{+A\lor B}\,(-\lor\mathrm{I})^2$$

3.4. Ancillary reductions

Ancillary reductions eliminate formulae that are consequences of *Smilean reductio* and premises of (Rejection). Because of the way the normalisation process takes place, we need only give them for atomic formulae. In what follows, then, α ranges over arbitrary atoms, and $\overline{\alpha}$ denotes the conjugate of α.[6] There are three cases to consider:

[6]The conjugate of a signed formula $+A$ is $-A$ and vice versa.

Case 1: One of the premises of (Rejection) is not the conclusion of *Smilean Reductio*. Suppose, without loss of generality, that it is the left one:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \alpha\end{array} \qquad \dfrac{\begin{array}{c}[\alpha]^1\\ \mathcal{D}_2\\ \bot\end{array}}{\overline{\alpha}}\,(\mathrm{SR})^1}{\bot}\,(\mathrm{Rejection}) \qquad\rightsquigarrow\qquad \begin{array}{c}\mathcal{D}_1\\ \alpha\\ \mathcal{D}_2\\ \bot\end{array}$$

Note that since α is an atom, this eliminates the ancillary maximal formula in question whilst introducing no further maximal formulae of any kind.

Case 2: Both premises of (Rejection) are the conclusion of *Smilean Reductio*, and at least one of the applications of *Smilean Reductio* discharges no premises of (Rejection). Suppose, without loss of generality, that the rightmost application is of this type:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \alpha\end{array} \qquad \dfrac{\begin{array}{c}[\alpha]^1\\ \mathcal{D}_2\\ \bot\end{array}}{\overline{\alpha}}\,(\mathrm{SR})^1}{\bot}\,(\mathrm{Rejection}) \qquad\rightsquigarrow\qquad \begin{array}{c}\mathcal{D}_1\\ \alpha\\ \mathcal{D}_2\\ \bot\end{array}$$

Note that again this reduces the number of ancillary maximal formulae and gives rise to no maximal formulae of any other type.

Case 3: Both premises of (Rejection) are conclusions of *Smilean reductio* and discharge some premise of (Rejection).

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{[\alpha]^1 \qquad \begin{array}{c}\mathcal{D}_0\\ \overline{\alpha}\end{array}}{\bot}\\ \mathcal{D}_1\\ \bot\end{array}}{\overline{\alpha}}\,(\mathrm{SR})^1 \qquad \begin{array}{c}\mathcal{D}_2\\ \alpha\end{array}}{\bot}\,(\mathrm{Rejection}) \qquad\rightsquigarrow\qquad \begin{array}{c}\dfrac{\begin{array}{c}\mathcal{D}_2\\ \alpha\end{array} \qquad \begin{array}{c}\mathcal{D}_0\\ \overline{\alpha}\end{array}}{\bot}\,(\mathrm{Rejection})\\ \mathcal{D}_1\\ \bot\end{array}$$

Suppose we apply this reduction to an ancillary maximal formula such that there are no ancillary maximal formulae above it or above a formula side connected with it. In the original derivation there may be further occurrences of α with discharge label 1 besides the one explicitly represented above. We also replace them with a copy of $\mathcal{D}_2$ ending in α. Those occurrences that were not premises of (Rejection) are unproblematic. Those that were, on the other hand, become new ancillary maximal formulae of the same complexity. By assumption, though, they are ancillary maximal formulae of the type covered in **Case 1**. We eliminate them as part of the current reduction step, and as a result the number of ancillary maximal formulae decreases, and we give rise to no maximal formulae of other kinds.

3.5. Normalisation and corollaries

THEOREM 3.2 (Normalisation). *If there is a derivation $\mathcal{D}$ of φ from Γ then there is a normal derivation $\mathcal{D}'$ of φ from $\Gamma' \subseteq \Gamma$.*

PROOF: To each derivation $\mathcal{D}$ we assign a coordination rank $(n, m) \in \mathbb{N}\times\mathbb{N}$, where n is the highest complexity of a maximal coordination formula, and m the number of maximal coordination formulae of maximal complexity. A derivation without maximal coordination formulae has rank $(0, 0)$, and coordination ranks are ordered lexicographically. We also assign it an operational rank $(j, k) \in \mathbb{N} \times \mathbb{N}$ defined analogously but with respect to maximal operational formulae, and order operational ranks with their own lexicographical order. The following is an effective procedure to normalise derivations:

1. Take a maximal coordination formula of the highest complexity such that there are no coordination formulae of the highest complexity above it or above a formula side-connected with it. Apply the appropriate reduction from Sections 3.2 and 3.3. The coordination rank strictly decreases. Thus, after a finite number of steps, our derivation has coordination rank (0,0).

2. Take a maximal operational formula of the highest complexity such that there are no maximal operational formulae of the highest complexity above it or above a formula side-connected with it. Apply the appropriate reduction from Section 3.1. The operational rank

strictly decreases, and the coordination rank stays $(0, 0)$. Thus, after a finite number of steps, our derivation has coordination and operational ranks (0,0).

3. Take an ancillary maximal formula (note that it must be atomic) such that that there are no ancillary maximal formulae above it or above a formula side-connected with it. Apply the appropriate reduction from Section 3.4. The number of ancillary maximal formulae goes down, and the coordination and operational ranks stay the same. After a finite number of steps, the derivation is in normal form. □

Definition 3.3. (Branch)
A branch π in a derivation $\mathcal{D}$ is a sequence $\varphi_1, ..., \varphi_n$ of occurrences of formulae or of $\bot$ such that: **(i)** φ_1 is a leaf (an assumption), discharged or not. **(ii)** φ_{i+1} stands immediately below φ_i. **(iii)** φ_n is either the conclusion of $\mathcal{D}$ or the first formula occurrence in the sequence that is the minor premise of $(+ \to \text{E})$, $(+\vee \text{ E})$ or $(-\wedge\text{E})$.

Lemma 3.4. *Every formula in a derivation belongs to some branch.*

Proof: By induction on derivations. □

The following theorem characterises the shape of normal derivations (see also Remark 3.6 for a comparison with Prawitz's normal form).

Theorem 3.5 (Shape of normal derivations). *Let $\mathcal{D}$ be a normal derivation, $\pi = \varphi_1, ..., \varphi_n$ a branch in it. Then there is a minimum formula φ_i dividing π into two (possibly empty) parts, the E part and the I-part, such that:*
(i)*Each φ_j in the E-part (i.e. $j < i$) is the major premise of an E-rule.*
(ii) *If $i \neq n$ then φ_i is a premise of (Rejection) or an I-rule.*
(iii) *Each φ_k in the I-part (i.e. $i < k$) is a premise of an I-rule, except φ_{i+1}, which may be a premise $\bot$ of Smilean reductio.*

Proof: Let $\pi = \varphi_1, ..., \varphi_n$ be a branch in a normal derivation $\mathcal{D}$. Then in π there are a) no applications of an E-rule after an I-rule, b) no applications of *Smilean reductio* after an I-rule, c) no applications of (Rejection) after an I-rule and d) no applications of an E-rule after *Smilean reductio*. I will prove a) as an example; b)-d) are proved analogously.

No applications of an E-rule after an I-rule: suppose for a contradiction that there are, let φ_k be the first consequence of an E-rule applied after an I-rule, consider φ_{k-1}. Since the derivation is normal, φ_{k-1} is not the consequence of an I-rule. It cannot be the consequence of *Smilean reductio* or (Rejection) either, as then we couldn't obtain φ_k from it through an E-rule. Thus, φ_{k-1} must be the consequence of an E-rule, contradicting the assumption that φ_k was first.

The remainder of the theorem is easy to prove: consider the last rule applied in π: if it is an E-rule, let $\varphi_i = \varphi_n$. If it is *Smilean reductio*, let $\varphi_i = \varphi_{n-2}$. If it is (Rejection), let $\varphi_i = \varphi_1$. Finally, if the last rule is an I-rule, let φ_i be the only formula occurrence in π that is a premise of (Rejection) – if there is one –, or else let φ_i be the first premise of an I-rule.

□

Remark 3.6. An alternative way of phrasing Theorem 3.5 is to say that a branch in a normal derivation consists of three (possibly empty) parts: an E-part, where every formula occurrence is the major premise of an E-rule, a C-part, where every formula occurrence is atomic and a premise of a coordination principle, and an I-part, where every formula occurrence is a premise of an I-rule. Branches in Prawitz's classical normal derivations (see [7]) consist of an E-part and an I-part, joined together by a (possibly empty) part where classical reductio is applied to an atom.

We can now obtain the subformula and separation properties as corollaries.

Definition 3.7. (Subformula)
Signed formula ψ is a subformula of signed formula φ if the unsigned part of ψ is a subformula (in the standard sense) of the unsigned part of φ. Thus, for example, all of $+p$, $-p$, $+q$, $-q$ are signed subformulae of $+p \to q$. Note that $\bot$ is not a formula but a punctuation sign.

Definition 3.8 (Order of a branch). A branch $\pi = \varphi_1, ..., \varphi_n$ in a derivation $\mathcal{D}$ is of order 0 if φ_n is the conclusion of $\mathcal{D}$, and of order $k+1$ if it ends on the minor premise of an E-rule the major premise of which belongs to a branch π' order k.

Corollary 3.9 (Subformula property). All the formulae that occur in a normal derivation of φ from Γ are subformulae of some $\gamma \in \Gamma$ or of φ.

Proof: By induction on the order of branches. Let $\pi = \varphi_1, ..., \varphi_n$ be a branch of order k and assume the result for branches of order $j < k$. We will think of the E, C and I-parts of a branch as defined in Remark 3.6. The result is obvious for the I-part: if $k = 0$ all formulae in it are subformulae of $\varphi_n = \varphi$. Similarly, if $k > 0$ then all formulae in the I-part are subformulae of φ_n, which is in its turn a subformula of the major premise ψ of an elimination rule that belongs to a branch of lower order. By inductive hypothesis ψ is itself subformula of some $\gamma \in \Gamma$ or of φ, and therefore so are all the formulae in the I-part.

It remains to show the result for the E and C-parts. Note that all remaining formulae are subformulae of φ_1, the first formula of the branch. Now, if φ_1 is an undischarged assumption the result follows trivially. If φ_1 is a discharged assumption, there are two cases to consider:

Case 1: If φ_1 is discharged by *Smilean reductio* then φ_1 must be an atom, and so the E-part of our branch π is empty. Moreover, the application of *Smilean reductio* in question concludes $\overline{\varphi}_1$, the conjugate of φ_1. Note that φ_1 is a subformula of $\overline{\varphi}_1$, and that $\overline{\varphi}_1$ must be a subformula of φ_n, the last formula of the branch. Thus, φ_1 is a subformula of φ_n. If the branch π is of order 0 this means that φ_1 is a subformula of the conclusion, and if π is of order > 0 then the result follows by inductive hypothesis.

Case 2: If φ_1 is discharged by an I-rule, then it is a subformula of the consequence φ_k of that application, and φ_k is in its turn a subformula of φ_n, the last formula in the branch. Once again, if the branch π is of order 0 this means that φ_1 is a subformula of the conclusion, and if π is of order > 0 then the result follows by inductive hypothesis. □

Corollary 3.10 (Separation property). In a normal derivation of φ from Γ only operational rules for connectives in φ and Γ (and perhaps coordination principles) are used.

Proof: Follows immediately from Corollary 3.9. □

4. Harmony and normalisation: $\mathbf{HB}_2$

The first two clauses of the definition of normal form for $\mathbf{HB}_2$ are identical to those of $\mathbf{HB}_1$. In other words, we require that for all normal derivations of $\mathbf{HB}_2$:

(i) No conclusion of an I-rule is a major premise of an E-rule.
(ii) Coordination principles are applied only to atoms.

The motivation behind them remains the same: **(i)** is taken from Prawitz, and **(ii)** is its analogue for bilateral systems, suggested by del Valle-Inclan and Schlöder's notion of harmony. This is, as before, enough to ensure that normal derivations satisfy the separation property. Once again, however, it is not enough to obtain the subformula property, as the following derivation shows:

$$\dfrac{\dfrac{+p \qquad -p}{+q}\,(\text{ex}) \qquad \dfrac{+p \qquad -p}{-q}\,(\text{ex})}{+r}\,(\text{ex})$$

In order to secure the subformula property we follow the same strategy as before: imposing constraints on the way coordination principles interact *with each other*. These constraints are given by clause **(iii)** of Definition 4.1.

Definition 4.1. (Normal form)
A derivation in $\mathbf{HB}_2$ is in normal form if in it: **(i)** No conclusion of an I-rule is a major premise of an E-rule. **(ii)** Coordination principles are applied only to atoms. **(iii)** (a) No conclusion of (ex) is a premise of (ex), (b) no application of (ex) has both premises discharged by (bem) and, (c) no conclusion of (bem) is a premise of (ex).

Formula occurrences that infringe clauses **(i)** and **(ii)** are called maximal operational formulae and maximal coordination formulae, respectively. Formula occurrences that infringe clause **(iii)** are called ancillary maximal formulae.

$\mathbf{HB}_1$ and $\mathbf{HB}_2$ share the same operational rules, so the reduction steps for operational maximal formulae are identical. The obvious similarity between the rules (Rejection) and (ex) means that the reduction steps to restrict (ex) to atomic premises are analogous to the steps restricting (Rejection) to atomic premises; I will omit this type of reduction as well, for reasons of space. The remaining reduction steps are as follows.

4.1. Reducing (ex) to atomic conclusions

Negation:

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ +A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ -A\end{matrix}}{+\neg B}\,(\text{ex}) \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ +A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ -A\end{matrix}}{-B}\,(\text{ex})}{+\neg B}\,(-\neg\ \text{I})$$

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ +A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ -A\end{matrix}}{-\neg B}\,(\text{ex}) \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ +A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ -A\end{matrix}}{+B}\,(\text{ex})}{-\neg B}\,(-\neg\ \text{I})$$

Implication:

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ +A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ -A\end{matrix}}{+B \to C}\,(\text{ex}) \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ +A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ -A\end{matrix}}{+C}\,(\text{ex})}{+B \to C}\,(+\to\ \text{I})^0$$

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ +A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ -A\end{matrix}}{-B \to C}\,(\text{ex}) \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ +A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ -A\end{matrix}}{+B}\,(\text{ex}) \qquad \dfrac{\begin{matrix}\mathcal{D}_1 \\ +A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ -A\end{matrix}}{-C}\,(\text{ex})}{-B \to C}\,(-\to\ \text{I})$$

Disjunction:

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ +A\end{matrix}\qquad\begin{matrix}\mathcal{D}_2\\ -A\end{matrix}}{+B\vee C}\,(\text{ex})\qquad\rightsquigarrow\qquad\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ +A\end{matrix}\qquad\begin{matrix}\mathcal{D}_2\\ -A\end{matrix}}{+C}\,(\text{ex})}{+B\vee C}\,(+\vee\text{ I})^0$$

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ +A\end{matrix}\qquad\begin{matrix}\mathcal{D}_2\\ -A\end{matrix}}{-B\vee C}\,(\text{ex})\qquad\rightsquigarrow\qquad\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ +A\end{matrix}\qquad\begin{matrix}\mathcal{D}_2\\ -A\end{matrix}}{-B}\,(\text{ex})\qquad\dfrac{\begin{matrix}\mathcal{D}_1\\ +A\end{matrix}\qquad\begin{matrix}\mathcal{D}_2\\ -A\end{matrix}}{-C}\,(\text{ex})}{-B\vee C}\,(-\vee\text{ I})$$

4.2. Reducing assumptions to atoms in (bem)

Negation:

$$\dfrac{\begin{matrix}[+\neg A]^1\\ \mathcal{D}_1\\ \varphi\end{matrix}\qquad\begin{matrix}[-\neg A]^1\\ \mathcal{D}_2\\ \varphi\end{matrix}}{\varphi}\,(\text{bem})^1\qquad\rightsquigarrow\qquad\dfrac{\begin{matrix}\dfrac{[-A]^1}{+\neg A}\\ \mathcal{D}_1\\ \varphi\end{matrix}\qquad\begin{matrix}\dfrac{[+A]^1}{-\neg A}\\ \mathcal{D}_2\\ \varphi\end{matrix}}{\varphi}\,(\text{bem})^1$$

Implication:

$$\dfrac{\begin{matrix}[+A\to B]^1\\ \mathcal{D}_1\\ \varphi\end{matrix}\qquad\begin{matrix}[-A\to B]^1\\ \mathcal{D}_2\\ \varphi\end{matrix}}{\varphi}\,(\text{bem})^1\qquad\rightsquigarrow$$

$$\dfrac{\begin{matrix}\dfrac{[+B]^3}{+A\to B}\\ \mathcal{D}_1\\ \varphi\end{matrix}\qquad\dfrac{\begin{matrix}\dfrac{[-B]^3\qquad[+A]^2}{-A\to B}\\ \mathcal{D}_2\\ \varphi\end{matrix}\qquad\begin{matrix}\dfrac{\dfrac{[-A]^2\qquad[+A]^1}{+B}\,(\text{ex})}{+A\to B}\,(+\to\text{ I})^1\\ \mathcal{D}_1\\ \varphi\end{matrix}}{\varphi}\,(\text{bem})^2}{\varphi}\,(\text{bem})^3$$

Disjunction:

$$\dfrac{\begin{array}{c}[+A\vee B]^1\\ \mathcal{D}_1\\ \varphi\end{array}\qquad\qquad \begin{array}{c}[-A\vee B]^1\\ \mathcal{D}_2\\ \varphi\end{array}}{\varphi}\,(\mathrm{bem})^1 \qquad\qquad \rightsquigarrow$$

$$\dfrac{\begin{array}{c}\dfrac{[+B]^3}{+A\vee B}\\ \mathcal{D}_1\\ \varphi\end{array}\qquad \dfrac{\begin{array}{c}\dfrac{[-B]^3 \qquad [-A]^2}{-A\vee B}\\ \mathcal{D}_2\\ \varphi\end{array}\qquad \begin{array}{c}\dfrac{\dfrac{[-A]^1 \qquad [+A]^2}{+B}\,(\mathrm{ex})}{+A\vee B}\,(+\vee\ \mathrm{I})^1\\ \mathcal{D}_1\\ \varphi\end{array}}{\varphi}\,(\mathrm{bem})^2}{\varphi}\,(\mathrm{bem})^3$$

4.3. Reducing conclusions to atoms in (bem)

Negation:

$$\dfrac{\begin{array}{c}[+A]^1\\ \mathcal{D}_1\\ +\neg B\end{array}\qquad \begin{array}{c}[-A]^1\\ \mathcal{D}_2\\ +\neg B\end{array}}{+\neg B}\,(\mathrm{bem})^1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\begin{array}{c}[+A]^1\\ \mathcal{D}_1\\ +\neg B\end{array}}{-B}\,(+\neg\ \mathrm{E})\qquad \dfrac{\begin{array}{c}[-A]^1\\ \mathcal{D}_2\\ +\neg B\end{array}}{-B}\,(+\neg\ \mathrm{E})}{-B}\,(\mathrm{bem})^1}{+\neg B}\,(+\neg\ \mathrm{I})$$

The case where $\varphi = -\neg B$ is analogous.

Implication:

$$\dfrac{\begin{matrix}[+A]^1\\ \mathcal{D}_1\\ +B\to C\end{matrix}\qquad\begin{matrix}[-A]^1\\ \mathcal{D}_2\\ +B\to C\end{matrix}}{+B\to C}\,(\text{bem})^1 \qquad\qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[+A]^1\\ \mathcal{D}_1\\ +B\to C\end{matrix}\qquad [+B]^2}{+C}\qquad\dfrac{\begin{matrix}[-A]^1\\ \mathcal{D}_2\\ +B\to C\end{matrix}\qquad [+B]^2}{+C}}{+C}\,(\text{bem})^1}{+B\to C}\,(+\to \mathrm{I})^2$$

$$\dfrac{\begin{matrix}[+A]^1\\ \mathcal{D}_1\\ -B\to C\end{matrix}\qquad\begin{matrix}[-A]^1\\ \mathcal{D}_2\\ -B\to C\end{matrix}}{-B\to C}\,(\text{bem})^1 \qquad\qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[+A]^1\\ \mathcal{D}_1\\ -B\to C\end{matrix}}{+B}\,(-\to\mathrm{E})\qquad\dfrac{\begin{matrix}[-A]^1\\ \mathcal{D}_2\\ -B\to C\end{matrix}}{+B}\,(-\to\mathrm{E})}{+B}\,(\text{bem})^1\qquad\dfrac{\dfrac{\begin{matrix}[+A]^2\\ \mathcal{D}_1\\ -B\to C\end{matrix}}{-C}\,(-\to\mathrm{E})\qquad\dfrac{\begin{matrix}[-A]^2\\ \mathcal{D}_2\\ -B\to C\end{matrix}}{-C}}{-C}\,(\text{bem})^2}{-B\to C}\,(-\to\mathrm{I})$$

Disjunction: Analogous to implication.

4.4. Ancillary reductions

As before, the α_i range over arbitrary *atoms*, and $\overline{\alpha_i}$ denotes the conjugate of α_i.
Clause **(iii)**(a):

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ +p\end{matrix}\qquad\begin{matrix}\mathcal{D}_2\\ -p\end{matrix}}{\alpha_1}\,(\mathrm{ex})\qquad\begin{matrix}\mathcal{D}_3\\ \overline{\alpha}_1\end{matrix}}{\alpha_2}\,(\mathrm{ex})\qquad\rightsquigarrow\qquad\dfrac{\begin{matrix}\mathcal{D}_1\\ +p\end{matrix}\qquad\begin{matrix}\mathcal{D}_2\\ -p\end{matrix}}{\alpha_2}\,(\mathrm{ex})$$

Clause **(iii)**(b):

$$(\mathrm{bem})^n\,\dfrac{\begin{matrix}\dfrac{\begin{matrix}\dfrac{[\alpha_1]^1\qquad[\overline{\alpha}_1]^n}{\alpha_2}\,(\mathrm{ex})\\ \mathcal{D}_1\\ \alpha_3\end{matrix}\qquad\begin{matrix}[\overline{\alpha}_1]^1\\ \mathcal{D}_2\\ \alpha_3\end{matrix}}{\alpha_3}\,(\mathrm{bem})^1\\ \mathcal{D}_3\\ \alpha_4\end{matrix}\qquad\begin{matrix}[\alpha_1]^n\\ \mathcal{D}_4\\ \alpha_4\end{matrix}}{\alpha_4}\qquad\rightsquigarrow\qquad\dfrac{\begin{matrix}[\overline{\alpha}_1]^1\\ \mathcal{D}_2\\ \alpha_3\\ \mathcal{D}_3\\ \alpha_4\end{matrix}\qquad\begin{matrix}[\alpha_1]^1\\ \mathcal{D}_4\\ \alpha_4\end{matrix}}{\alpha_4}\,(\mathrm{bem})^1$$

Note that in this last reduction we have assumed that the left premise of the application of (ex) is discharged before the right one. This is unimportant: if it is the other way around, the appropriate reduction is analogous.

Clause **(iii)**(c):

$$(\mathrm{ex})\,\dfrac{\dfrac{\begin{matrix}[+p]^1\\ \mathcal{D}_1\\ \alpha_1\end{matrix}\qquad\begin{matrix}[-p]^1\\ \mathcal{D}_2\\ \alpha_1\end{matrix}}{\alpha_1}\,(\mathrm{bem})^1\qquad\begin{matrix}\mathcal{D}_3\\ \overline{\alpha}_1\end{matrix}}{\alpha_2}\qquad\rightsquigarrow\qquad\dfrac{\dfrac{\begin{matrix}[+p]^1\\ \mathcal{D}_1\\ \alpha_1\end{matrix}\qquad\begin{matrix}\mathcal{D}_3\\ \overline{\alpha}_1\end{matrix}}{\alpha_2}\,(\mathrm{ex})\qquad\dfrac{\begin{matrix}[-p]^1\\ \mathcal{D}_2\\ \alpha_1\end{matrix}\qquad\begin{matrix}\mathcal{D}_3\\ \overline{\alpha}_1\end{matrix}}{\alpha_2}\,(\mathrm{ex})}{\alpha_2}\,(\mathrm{bem})^1$$

Applications of (ex) like the one above on the left, where at least one of the premises is a conclusion of (bem), are called **peaks**. The **size** of a peak is the sum of the length of the maximal segments that the premises of (ex) belong to (if a premise is not part of a maximal segment, we assign it length 0). In the normalisation process we will assign to each derivation a **peak**

rank (j, k), where j is the greatest size of a peak in the derivation, k the number of peaks of greatest size. The reader can check that the reduction above, when applied to a maximal segment such that there are no longer maximal segments above it, side connected with it, or above a formula side connected with it, strictly reduces the peak rank of a derivation.

4.5. Normalisation and corollaries

Definition 4.2 (Segment). A segment σ in a branch π is a sequence of formula occurrences $\sigma_1, ..., \sigma_n$ in π such that: **(i)** σ_1 is *not* the conclusion of an application of (bem). **(ii)** Each σ_i for $i < n$ is a premise of (bem), and σ_{i+1} stands immediately below σ_i. **(iii)** σ_n is *not* a premise of an application of (bem).

Definition 4.2 entails that all the elements of a segment are occurrences of the same formula. The **length of a segment** is the number of formula occurrences in it. A segment is called **maximal** if it ends in an application of (ex). This means that maximal coordination formulae that infringe clause **(iii)**(c) of Definition 4.1 are always final formula occurrences in maximal segments of length ≥ 1, and maximal coordination formulae that infringe clauses **(iii)**(a) and **(iii)**(b) are always maximal segments of length 1. There are no maximal segments of other types.

Lemma 4.3. *Every branch can be uniquely divided into consecutive segments.*

Proof: By induction on the length of branches. □

Theorem 4.4 (Normalisation). *If there is a derivation $\mathcal{D}$ of φ from Γ then there is a normal derivation $\mathcal{D}'$ of φ from $\Gamma' \subseteq \Gamma$.*

Proof: Analogous to the previous proof of normalisation. Derivations are assigned a coordination and an operational rank, defined as before. We apply first the coordination reductions (Sections 4.1–4.3) and then the operational reductions (Section 3.1), starting always from maximal formulae of maximal complexity such that there are no maximal formulae of maximal complexity above them or above a formula side connected with them. Once a derivation has no coordination or operational maximal formulae we assign it a peak rank, as defined at the end of Section 4.4, and apply the reduction for ancillary formulas of type **(iii)**(c) as indicated there. Once there are no peaks left, the only remaining maximal formulae are those that infringe clauses **(iii)**(a) and **(iii)**(b). They can be eliminated in any order using the appropriate reduction from Section 4.4. □

THEOREM 4.5 (Shape of normal derivations). *Let $\mathcal{D}$ be a derivation in normal form, π a branch in $\mathcal{D}$, and let $\sigma_1, ...\sigma_n$ be the segments in π. Then there is a segment σ_i in π, called the minimum segment, which separates /pi into two (possibly empty) parts, the E-part and the I-part, with the properties:*

1. *For each σ_j in the E-part (i.e. $j < i$), σ_j is a major premise of an E-rule, except possibly σ_{i-1}, which may be a premise of (ex).*
2. *If $i \neq n$, then each formula in the segment σ_i is a premise of (bem) except the last one, which may be a premise of an I-rule.*
3. *For each σ_j in the I-part (i.e. $i < j < n$), σ_j is a premise of an I-rule.*

PROOF: It is easy to see that, in a branch $\pi = \varphi_1, ..., \varphi_n$ of a normal derivation, no formula occurrences that are premises of an Introduction rule precede formula occurrences that are major premises of an Elimination rule, (bem) or (ex), no formula occurrences that are premises of (bem) precede formula occurrences that are premises of (ex) or major premises of an E-rule, and no formula occurrences that are premises of (ex) precede formula occurrences that are major premises of an E-rule or (ex). Now:

If there is no formula occurrence that is a premise of an I-rule or (bem), let $\sigma_i = \varphi_n$. If there is a formula occurrence that is a premise (bem), let φ_i be the first such formula, and let σ_i be the segment starting at φ_i. Finally, if there is no formula occurrence that is a premise (bem), but there is a formula occurrence that is a premise of an I-rule, let φ_i be the first such formula, and let $\sigma_i = \varphi_i$. □

Remark 4.6. An alternative way of phrasing Theorem 4.5 is to say that a branch in a normal derivation consists of three (possibly empty) parts: an E-part, where every formula occurrence is the major premise of an E-rule, a C-part, where every formula occurrence is a premise of a coordination principle – and within which (ex) is applied before (bem) – and an I-part, where every formula occurrence is a premise of an I-rule.

Corollary 4.7 (Subformula property). All the formulae that occur in a normal derivation of φ from Γ are subformulae of some $\gamma \in \Gamma$ or of φ.

Proof: By induction on the order of branches. Let $\pi = \sigma_1, ...\sigma_n$ be a branch of order p, let σ_i be its minimum segment, and assume the result for branches of lower order. Consider first all σ_j with $i \leq j \leq n$. All such formulae are subformulae of φ_n, the formula in the last segment σ_n of the branch. If the branch in question is of order 0 the result immediately follows. If the branch is of order > 0 then φ_n is the minor premise of an application of an E-rule, the major premise ψ of which belongs to a branch of order $p - 1$. But by induction hypothesis the result holds for ψ, and φ_n is a subformula of ψ, so the result follows.

It remains to account for all the σ_j with $j < i$. Note that all such formulae are subformulae of φ_1, the first formula of the branch. If φ_1 is an undischarged assumption the result immediately follows. Similarly, if φ_1 is discharged by an application of an I-rule, then it is a subformula of some formula in an I-part, and the result follows by the above. Finally, suppose that φ_1, is discharged by an application of (bem). Now, φ_1 cannot be the major premise of an elimination rule, since it is an atom. If it is the minor premise of an E-rule, or a premise of an I-rule or (bem), then there are no σ_j with $j < i$ and we are done. The only remaining possibility is that φ_1 is a premise of (ex). Then φ_1 is the only formula before the minimum segment σ_i (in other words, φ_1 is the only formula we still need to account for). Now, φ_1 is a subformula of the other premise $\overline{\varphi}_1$ of the application of (ex) in question, and $\overline{\varphi}_1$ cannot be discharged by (bem). Moreover, $\overline{\varphi}_1$ belongs to a branch of the same order as π. If $\overline{\varphi}_1$ is undischarged, or discharged by a I-rule, the result immediately follows. If it is a consequence of an E-rule, then it is a subformula of the initial formula ψ of its branch. But then ψ is not atomic, and so can only be undischarged or discharged by an I-rule. In either case, the result follows. □

Corollary 4.8 (Separation property). In a normal derivation of φ from Γ only operational rules for connectives in φ and Γ (and perhaps coordination principles) are applied.

Proof: Follows immediately from the previous corollary. □

5. Comparison with other normalisation results

In this section I will briefly compare the present normalisation results with those obtained by Nils Kürbis [6] and Marcello D'Agostino, Dov Gabbay, and Sanjay Modgyl [1], so as to outline the similarities and differences between them.

5.1. $\mathbf{HB}_1$ and Kürbis-normal form

Kürbis proves his normalisation result for Rumfitt's original calculus,[7] which makes the comparison with normal form for $\mathbf{HB}_1$ straightforward. The respective definitions of normal form are:

Kürbis-normal form:	**$\mathbf{HB}_1$-normal form:**
(a) No conclusion of an I-rule is a major premise of an E-rule.	(i) No conclusion of an I-rule is a major premise of an E-rule.
(b) No conclusion of *Smilean reductio* is a major premise of an E-rule.	(ii) Coordination principles are applied only to atoms.
(c) No conclusion of an I-rule is a premise of an application of (Rejection) the other premise of which is also the conclusion of an I-rule.	
(d) No conclusion of *Smilean reductio* is a premise of (Rejection).	(iii) No conclusion of *Smilean reductio* is a premise of (Rejection).
(e) There are no maximal segments.	

Clause (a) of Kürbis-normal form is identical to clause (i) of $\mathbf{HB}_1$-normal form, and the same goes for clauses (d) and (iii). The correlate of clauses (b) and (c) of Kürbis-normal form is clause (ii). Crucially, though, (ii) is strictly stronger that (b) and (c) combined: all derivations that satisfy (ii) satisfy (b) and (c), but the converse does not hold. The segments referred to in clause (e) are defined as usual: sequences of occurrences of the same

[7] That is, the calculus comprising the operational rules without del Valle-Inclan and Schlöder's modifications, plus (Rejection) and *Smilean reductio* as coordination principles.

formula that end in a maximal formula. Because of the modified operational rules of $\mathbf{HB}_1$, maximal segments simply cannot arise in normal derivations. Thus, clause (e) has no correlate in $\mathbf{HB}_1$-normal form. It follows that $\mathbf{HB}_1$-normal form is stronger than Kürbis-normal form, in the sense that all derivations in $\mathbf{HB}_1$-normal form are Kürbis-normal, but the converse does not hold. Rumfitt's calculus, for instance, can be Kürbis-normalised but not $\mathbf{HB}_1$-normalised.

5.2. $\mathbf{HB}_2$ and C-intelim normal form

Marcello D'Agostino, Dov Gabbay, and Sanjay Modgyl prove their normalisation result for a calculus they call C-intelim.[8] The crucial difference between C-intelim and $\mathbf{HB}_2$ is that no operational rule of C-intelim discharges any premises. More precisely, their rules for disjunction are Rumfitt's $(-\vee \text{ I})$, $(-\vee \text{ E})$, $(+\vee \text{ I})$ plus the following two:

$$\frac{+A \vee B \qquad -A}{+B} \qquad\qquad \frac{+A \vee B \qquad -B}{+A}$$

Their rules for conjunction are Rumfitt's $(+\wedge \text{ I})$, $(+\wedge \text{ E})$, $(-\wedge \text{ I})$ and:

$$\frac{-A \wedge B \qquad +A}{-B} \qquad\qquad \frac{-A \wedge B \qquad +B}{-A}$$

And their rules for conditionals are Rumfitt's $(- \to \text{ I})$, $(- \to \text{ E})$, $(+ \to \text{ E})$ and:

$$\frac{-A}{+A \to B} \qquad\qquad \frac{+B}{+A \to B} \qquad\qquad \frac{+A \to B \qquad -B}{-A}$$

The coordination principles in C-intelim are essentially *Explosion* and *Bilateral Excluded Middle*, but there is an additional consideration to keep in mind. In C-intelim *Explosion* is reformulated as two distinct rules, namely:

$$\frac{-A \qquad +A}{\perp} \qquad\qquad \frac{\perp}{\varphi}$$

with the proviso that '$\perp$' can only occur in the context of these rules, as a punctuation sign [1, p. 302]. Clearly, this makes the difference between (ex)

8 They present two versions of C-intelim: a bilateral version and a unilateral one, which they regard as a 'practically convenient translation of the rules for signed formulae into an ordinary logical language' ([1], p. 303-4). Here I will only consider the bilateral formulation of the calculus.

and their two-rule combination strictly notational. In order to simplify the comparison with $\mathbf{HB}_2$, then, I will take C-intelim to contain *Explosion* in its single-rule presentation (ex). Nothing substantial hinges on this.

We can finally compare both notions of normal form. Normal derivations in C-intelim have the following shape:

$$
\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ \varphi\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ \varphi\end{matrix}}{\begin{matrix}\varphi \\ \mathcal{T}_1 \\ \varphi\end{matrix}}\,(\text{bem}) \qquad \begin{matrix}\cdots \\ \\ \cdots\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_{n-1} \\ \varphi\end{matrix} \qquad \begin{matrix}\mathcal{D}_n \\ \varphi\end{matrix}}{\begin{matrix}\varphi \\ \mathcal{T}_k \\ \varphi\end{matrix}}\,(\text{bem})}{\varphi}\,(\text{bem})
$$

where in the $\mathcal{D}_i$ only operational rules are used, except possibly at the last step, which may be an application of *Explosion*, and the (possibly empty) $\mathcal{T}_i$ consist exclusively of applications of *Bilateral Excludded Middle.* Moreover, in normal derivations the assumptions discharged by (bem) are always subformulae of undischarged premises of the derivation or of its conclusion.

It is obvious that normal $\mathbf{HB}_2$ derivations need not be of this form. More importantly, derivations in $\mathbf{HB}_2$ cannot, in general, be put in C-intelim normal form. The reason is that certain operational rules of $\mathbf{HB}_2$ discharge premisses, which means that it is sometimes unavoidable to use them after an application of *Explosion*, as in the ($\mathbf{HB}_2$-normal) derivation below:

$$
\dfrac{\dfrac{\dfrac{+\neg p}{-p} \qquad [+p]^1}{+q}\,(\text{ex})}{+p \to q}\,(+\to \mathrm{I})^1
$$

Conversely, derivations in C-intelim cannot in general be put in $\mathbf{HB}_2$-normal form. This is due to the fact that several operational rules of C-intelim do not preserve the coordination principle of *Bilateral Excluded Middle*, and hence are not harmonious in del Valle-Inclan and Schlöder's sense. The following, for instance, is a C-intelim normal derivation where (bem) is applied to complex formulae in a way that cannot be eliminated:

$$\dfrac{\dfrac{[+p]^1}{+p \vee \neg p} \qquad \dfrac{\dfrac{[-p]^1}{+\neg p}}{+p \vee \neg p}}{+p \vee \neg p}\,(\text{bem})^1$$

In summary: C-intelim normal form and $\mathbf{HB}_2$-normal form are neither stronger nor weaker than each other.

Of course, normal forms for different but related calculi need not coincide on every point, so the fact that Rumfitt's calculus and C-intelim do not $\mathbf{HB}_1$ and $\mathbf{HB}_2$-normalise (respectively) is not particularly surprising. Still, given the close connection between the present notions of normal form and bilateral harmony, this can be seen as a symptom of the underlying disharmony of these calculi. Conversely, the results shown by Kürbis and D'Agostino, Gabbay and Modgyl show that, despite their disharmony, C-intelim and Rumfitt's calculus are relatively well-behaved. This emphasises the fact that del Valle-Inclan and Schlöder's notion of bilateral harmony rules out more than the glaring problems raised by connectives like tonk and bink.[9]

6. Concluding remarks

The idea that the operational rules for each connective should be 'balanced' underlies most approaches to proof-theoretic harmony. This idea has a correlate in normalisation proofs, in the requirement that formula occurrences that are the consequence of an introduction rule and the major premise of an elimination should be removed. Del Valle-Inclan and Schlöder's bilateral criterion of harmony suggests a similar requirement for bilateral systems, namely that normal proofs should apply coordination principles to atomic formulae only. These two requirements are enough for a weak notion of normality that remains stable across calculi $\mathbf{HB}_1$ and $\mathbf{HB}_2$, and which guarantees the separation but not the subformula property. In order to guarantee the subformula property a third kind of constraint, regulating how coordination principles are allowed to interact with each other, is needed. These constraints vary across $\mathbf{HB}_1$ and $\mathbf{HB}_2$, as the two calculi

[9]Thanks to an anonymous referee for prompting me to say more about this.

have different coordination principles. The resulting notion of normal form is strictly stronger than Kürbis-normal form in the case of $\mathbf{HB}_1$, and neither stronger nor weaker than C-intelim normal form in the case of $\mathbf{HB}_2$.

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Pedro del Valle-Inclan
Scuola Normale Superiore
Department of Philosophy
56126, Piazza dei Cavalieri 7
Pisa, Italy
e-mail: pedro.delvalleinclan@sns.it