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# EQUIVALENTIAL ALGEBRAS WITH CONJUNCTION ON DENSE ELEMENTS

**Abstract**

We study the variety generated by the three-element equivalential algebra with conjunction on the dense elements. We prove the representation theorem which let us construct the free algebras in this variety.

*Keywords*: Intuitionistic logic, Fregean varieties, equivalential algebras, dense elements.

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## 1. Introduction

According to [6], there are only finitely many polynomial clones on a finite algebra which generates a congruence permutable Fregean variety. As we will show in the paper, if a three-element algebra **A** generates a congruence permutable Fregean variety, then the universe of **A** with the natural order is a chain. Moreover, also the lattice of congruences on **A** is a three-element chain. It is known that congruence permutable Fregean varieties are congruence modular, so we can consider in this case the commutator operation. By [6, Corollary 2.8], due to the behavior of the commutator operation on a three-element algebra, we can distinguish four polynomially nonequivalent algebras, that generate congruence permutable Fregean varieties.

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Two of them are well known: the three-element equivalential algebra and the three-element Brouwerian semilattice. The equivalential algebras are solvable, so they are of type 2 ([6, p. 606]) in the sense of Tame Congruence Theory of Hobby and McKenzie [4]. However, the Brouwerian semilattices are congruence distributive and so they are of type 3. Equivalential algebras and Brouwerian semilattices have already been carefully studied, both when it comes to the construction of the $n$-generated free algebras, as well as the cardinality of these algebras for small $n$ and for some subvarieties, see [8, 19, 14, 15] for the equivalential algebras and [9] for the Brouwerian semilattices.

In the other two cases we are dealing with a mixed type. In the first case, we have type 3 at the top of congruence lattice and type 2 at its bottom, see Figure 1. An example of algebra, which meets these conditions is the three-element equivalential algebra with conjunction on the regular elements. The variety generated by this algebra was investigated in [11], where its properties, the representation theorem, the construction of the free algebra and the free spectrum were given.

The aim of this paper is to study the variety generated by the three-element algebra, in which the commutator operation behaves in the opposite way: type 2 is at the top of congruence lattice and type 3 at its bottom. Such structure is the subreduct of the three-element Heyting algebra, with the equivalence operation and the second binary operation which is conjunction on the dense elements.

Both the dense elements as well as the regular elements play an important role in the study of the relation between classical and intuitionistic logic. They appear indirectly in the Glivenko theorem according to which a formula $\varphi$ is a tautology of classical propositional calculus iff its double negation (i.e. the regularization of $\varphi$) is a tautology of intuitionistic propositional calculus. An algebraic version of this theorem refers directly to dense elements: we divide a Heyting algebra by the filter of all dense elements obtaining a Boolean algebra [12, p. 132].

## 2. Preliminary

Let $\mathbf{A}$ be an algebra. We say that $\mu \in \operatorname{Con}\mathbf{A}$ is **completely meet-irreducible** if $\mu \neq A^2$ and for any family $\{\mu_i : i \in I\} \subseteq \operatorname{Con}\mathbf{A}$ such that $\mu = \bigcap_{i\in I}\mu_i$, we have $\mu = \mu_i$ for some $i \in I$. If $\mu$ is completely meet-irreducible, then there exists the unique cover of $\mu$ in Con $\mathbf{A}$, denoted by $\mu^+$. We will denote by Cm($\mathbf{A}$) the set of all completely meet-irreducible congruences on $\mathbf{A}$. Similarly, we can define a **completely join-irreducible** congruence $\nu$ and the unique subcover of $\nu$ in Con $\mathbf{A}$, denoted by $\nu^-$. Let $\Theta(a, b)$ denote the congruence generated by $(a, b)$.

Now, we will recall the most important facts related to the concept of the commutator. At the beginning we need the following definition:

Definition 2.1 ([10, p. 252]). Let $\alpha, \beta, \eta$ be congruences of an algebra $\mathbf{A}$. We say that $\alpha$ **centralizes** $\beta$ modulo $\eta$, written: $C(\alpha, \beta; \eta)$, iff for all $n \geq 1$, and for every: $t \in \mathrm{Clo}_{n+1} A$, $(a, b) \in \alpha$ and $(c_1, d_1), \ldots, (c_n, d_n) \in \beta$ we have:

$$t(a, c_1, \ldots, c_n) \equiv_\eta t(a, d_1, \ldots, d_n) \text{ iff } t(b, c_1, \ldots, c_n) \equiv_\eta t(b, d_1, \ldots, d_n).$$

Definition 2.2 ([10, p. 252]). For congruences $\alpha$ and $\beta$ of $\mathbf{A} \in \mathcal{V}$, where $\mathcal{V}$ is a congruence modular variety, we define their **commutator**, denoted $[\alpha, \beta]$, to be the smallest congruence $\eta$ of $\mathbf{A}$ for which $\alpha$ centralizes $\beta$ modulo $\eta$, i. e., $\eta = \bigwedge\{\phi : C(\alpha, \beta; \phi)\}$.

Definition 2.3 ([2, p. 35, 47]). Let $\mathbf{A} \in \mathcal{V}$, where $\mathcal{V}$ is a congruence modular variety, $\alpha, \beta \in \mathrm{Con}\,\mathbf{A}$ and $\alpha \leq \beta$. Then:

1. $\beta$ is called **Abelian over** $\alpha$ if $[\beta, \beta] \leq \alpha$,
2. $\beta$ is called **Abelian** if $[\beta, \beta] = 0_{\mathbf{A}}$,
3. $\mathbf{A}$ is called **Abelian** if $[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$.

We say that an algebra $\mathbf{A}$ satisfies the condition (**C1**) if $\alpha \wedge [\beta, \beta] = [\alpha \wedge \beta, \beta]$ for all $\alpha, \beta \in \mathrm{Con}\,\mathbf{A}$.

*Remark* 2.4 ([5, p. 49]). In congruence modular varieties the condition (**C1**) gives $[\alpha, \beta] = (\alpha \wedge [\beta, \beta]) \vee (\beta \wedge [\alpha, \alpha])$, for $\alpha, \beta \in \mathrm{Con}\,\mathbf{A}$, so the commutator operation on congruences of $\mathbf{A}$ is uniquely determined by the diagonal, i. e., by elements of the form $[\alpha, \alpha]$.

If $\mathbf{A} \in \mathcal{V}$ and $\mathcal{V}$ is a congruence modular variety, we can define the following notion:

Definition 2.5 ([10, p. 252]). The **centralizer** of $\beta$ modulo $\alpha$, denoted $(\alpha : \beta)$, is the largest congruence $\gamma$ of $\mathbf{A}$ such that $\gamma$ centralizes $\beta$ modulo $\alpha$, i. e., $\gamma = \bigvee\{\phi : C(\phi, \beta; \alpha)\}$.

Now, we give basic information about Fregean varieties.

Definition 2.6 ([6, p. 597]). An algebra **A** with a distinguished constant term 1 is called **Fregean** if **A** is:

1. **1-regular**, i. e., $1/\alpha = 1/\beta$ implies $\alpha = \beta$ for all $\alpha, \beta \in \operatorname{Con} \mathbf{A}$,
2. **congruence orderable**, i. e., $\Theta_{\mathbf{A}}(1, a) = \Theta_{\mathbf{A}}(1, b)$ implies $a = b$ for all $a, b \in A$.

A variety $\mathcal{V}$ is said to be Fregean if all its algebras are Fregean. Natural examples of Fregean varieties are: equivalential algebras, Boolean algebras, Heyting algebras, Brouwerian semillatices or Hilbert algebras. Fregean varieties are closely related with the Fregean logics, see [1].

Congruence orderability allows us to introduce a natural partial order on the universe of **A** in the following way: $a \leq b$ iff $\Theta_{\mathbf{A}}(1, b) \subseteq \Theta_{\mathbf{A}}(1, a)$. Clearly, 1 is the greatest element in this order. From 1-regularity it follows that the Fregean varieties are congruence modular, see [3].

Next, we recall an important theorem, which characterizes subdirectly irreducible algebras in Fregean varieties.

Proposition 2.7 ([16, Proposition 3.1], [6, Lemma 2.1]). Let **A** be an algebra from a Fregean variety $\mathcal{V}$. Then **A** is subdireclty irreducible iff there is the largest non-unit element $*$ in $A$. Moreover, the monolith $\mu$ of **A** has the form $1/\mu = \{*, 1\}$ and all other cosets with respects to $\mu$ are one-element.

The Fregean varieties meet the condition (**C1**). Moreover, they satisfy the stronger condition (**SC1**):

Definition 2.8 ([6, p. 602]). If $\mu$ is the monolith of a subdirectly irreducible algebra **A** from a Fregean variety then the centralizer $(0 : \mu)$ does not exceed $\mu$.

Definition 2.9. An **equivalential algebra** is an algebra $(A, \leftrightarrow, 1)$ of type $(2, 0)$ that is a subreduct of a Heyting algebra with the binary operation $\leftrightarrow$ given by $x \leftrightarrow y := (x \to y) \wedge (y \to x)$.

In this paper, we adopt the convention of associating to the left and ignoring (or replacing with "$\cdot$") the symbol of equivalence operation. In 1975 J. K. Kabziński and A. Wroński proved that the class **E** of all equivalential algebras is equationally definable by identities: $xxy = y$, $xyzz = (xz)(yz)$, $(xy)(xzz)(xzz) = xy$, and so it forms a variety [8].

We know from [6, p. 598] that $\mathbf{E}$ is congruence permutable. Moreover, the following theorem is true:

THEOREM 2.10 ([6, Theorem 3.8]). *Let $\mathcal{V}$ be a congruence permutable Fregean variety. Then there exists a binary term $\leftrightarrow$ such that for every $\mathbf{A} \in \mathcal{V}$:*

1. $(A, \leftrightarrow, 1)$ *is an equivalential algebra;*
2. $\leftrightarrow$ *is a principal congruence term of* $\mathbf{A}$*, i. e.,* $(a, b) \in \alpha$ *iff* $(1, a \leftrightarrow b) \in \alpha$ *for every* $\alpha \in \operatorname{Con} \mathbf{A}$.

If $\mathcal{V}$ is a congruence permutable Fregean variety and $\mathbf{A} \in \mathcal{V}$, then we will denote an equivalential reduct of $\mathbf{A}$ by $\mathbf{A}^e$.

## 3. The clones of polynomials of a three-element algebra, which generates a congruence permutable Fregean variety

It is known that there exist only two polynomially nonequivalent algebras defined on a two-element set and generating a congruence permutable Fregean variety [6, p. 640]. We examine an analogous situation, but for a three-element set. The first question concerns the number of such polynomially nonequivalent algebras. By Theorem 2.10, for every algebra $\mathbf{A}$ from a congruence permutable Fregean variety there is a binary term $\leftrightarrow$ such as $\mathbf{A}^e$ is an equivalential algebra. In order to answer our question we first need to consider a three-element algebra $\mathbf{A}$ with a universe $\{1, a, b\}$, with the equivalence operation $\leftrightarrow$ and a constant term 1, which is the greatest element in $\mathbf{A}$ in the natural order.

PROPOSITION 3.1. Let $\mathbf{A}$ generate a congruence permutable Fregean variety with a constant term 1 and let $|A| = 3$. Then:

1. $A$ with the natural order is a chain,
2. $(\operatorname{Con} \mathbf{A}, \vee, \wedge)$ with the order $\subseteq$ is a three-element chain

PROOF: (1) Let $A = \{1, a, b\}$. Without loss of generality we can assume that $a \leftrightarrow b = a$, since otherwise (i.e. $a \leftrightarrow b = b$) the situation would be analogous. From Theorem 2.10 we have $\Theta(1, a \leftrightarrow b) = \Theta(a, b)$. Thus: $\Theta(1, a) = \Theta(1, a \leftrightarrow b) = \Theta(a, b)$. As $\Theta(1, b) \subseteq \Theta(1, a)$, so from a congruence orderability it follows that $a \leq b$ and consequently $a < b < 1$.

(2) Similarly, from a congruence orderability and inequalities $a < b < 1$ we get: $0_{\mathbf{A}} = \Theta(1, 1) \subsetneq \Theta(1, b) \subsetneq \Theta(1, a) = 1_{\mathbf{A}}$. Thus: $0_{\mathbf{A}} < \Theta(1, b) < 1_{\mathbf{A}}$. This completes the proof because in $\operatorname{Con}\mathbf{A}$ there are only principal congruences. $\square$

Since $\{1, a, b\}$ with the natural order forms a chain, thus we adopt the convention that the smallest element in a three-element chain will be denoted by 0, and the middle element by $*$. We conclude from Proposition 2.7 that an algebra $\mathbf{A}$, which fulfills the assumptions of Proposition 3.1, is a subdirectly irreducible with the monolith $\Theta(1, *)$. Note also, that if a three-element algebra $\mathbf{A}$ comes from a congruence permutable Fregean variety, then $\Theta(x, y) = \Theta^e(x, y)$, for $x, y \in A$.

By [6, Corollary 2.8], the clone of polynomials of a finite algebra from a congruence permutable Fregean variety is uniquely determined by its congruence lattice expanded by the commutator operation, i. e., by the structure $\operatorname{Concom}(\mathbf{A}) := (\operatorname{Con}\mathbf{A}; \wedge, \vee, [\cdot,\cdot])$. Thus, the number of clones of polynomials of $\mathbf{A}$ depends on the behaviour of the commutator operation on a three-element lattice of congruences. There are four such possibilities, shown in the figure below.

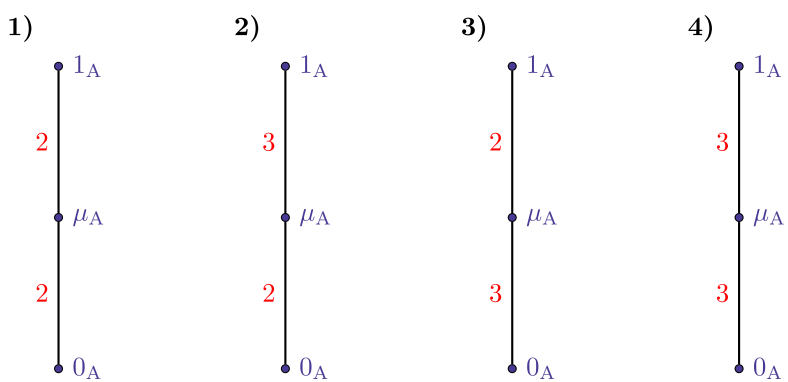


**Figure 1.**

The number 2 used in the figure means that a congruence $\alpha$ above this number is Abelian over $\alpha^-$, where $\alpha^-$ denotes the unique subcover of $\alpha$, i. e., $[\alpha, \alpha] = \alpha^-$. On the other hand, the number 3 means that a congruence $\alpha$ above this number fullfils: $[\alpha, \alpha] = \alpha$. Note that it follows from the condition (**SC1**) that the equality $[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$ is not possible, because it would lead to a contradiction, i. e., $(0_{\mathbf{A}} : \mu_{\mathbf{A}}) = 1_{\mathbf{A}}$.

An algebra, in which the commutator behaves as in the first case is the three-element equivalential algebra, whereas an algebra, in which the commutator behaves as in the fourth case is the three-element Brouwerian semilattice. An example corresponding to the second case is a three-element equivalential algebra with conjunction on the regular elements, described in [11]. In this article we will give an example of an algebra, in which the commutator behaves as in the third case.

## 4. Equivalential algebras with conjunction on the dense elements

In Heyting algebras we can consider both the dense elements and the regular elements. An element $x$ is called: **regular** if $(x \to 0) \to 0 = x$, **dense** if $(x \to 0) \to 0 = 1$. The Glivenko theorem mentioned earlier, explains their role in studying of the reducts of the intuitionistic logics. To defined them in Heyting algebras we use the constant 0. In equivalential algebras we can define the regular and dense elements without using this constant. In this situation we say that an element $x \in A$ is regular if $xyy = x$ for all $y \in A$, and it is dense if there is a finite subset $\{y_1, y_2, \ldots, y_n\} \subseteq A$ such that $xy_1y_1y_2y_2 \ldots y_ny_n = 1$. If the equivalential algebra $\mathbf{A}$ is the reduct of the Heyting algebra, then both definitions coincide.

In Heyting algebras we can define an operation of the conjuction on the dense elements. Let us consider a subreduct of the Heyting algebra with the constant 1 and with two binary operation. The first is the equivalence operation, providing the congruence permutability, while the second operation is the conjuction on the dense elements. We will also limit our considerations to the three-element subreduct of the Heyting algebra. From Proposition 3.1 we know that the universe of this algebra with the natural order forms a chain. Finally, we get the following definition.

DEFINITION 4.1. An **equivalential algebra with conjunction on the dense elements** is an algebra $\mathbf{D} := (\{0, *, 1\}, \cdot, d, 1)$ of type $(2, 2, 0)$, which is the reduct of the three-element Heyting algebra $\mathbf{H} = (\{0, *, 1\}, \wedge, \vee, \to, 0, 1)$ with an order: $0 < * < 1$, the constant 1, the equivalence operation $\cdot$ such that $x \cdot y := (x \to y) \wedge (y \to x)$, and an additional binary operation $d$ such that $d(x, y) := x00x \wedge y00y$.

Note that $\mathbf{A}^e$ is an equivalential algebra and $d$ is a binary commutative operation presented in the table below (on the right):

| $\cdot$ | 1 | $*$ | 0 |
|---|---|---|---|
| 1 | 1 | $*$ | 0 |
| $*$ | $*$ | 1 | 0 |
| 0 | 0 | 0 | 1 |

| $d$ | 1 | $*$ | 0 |
|---|---|---|---|
| 1 | 1 | $*$ | 1 |
| $*$ | $*$ | $*$ | $*$ |
| 0 | 1 | $*$ | 1 |

We denote by $\mathcal{V}(\mathbf{D})$ the variety generated by $\mathbf{D}$. It is easy to see, that $\mathbf{D}$ is a subdirectly irreducible Fregean algebra with the monolith denoted by $\mu_{\mathbf{D}}$. Moreover, $\operatorname{Con}\mathbf{D} = \{0_{\mathbf{D}}, \mu_D, 1_{\mathbf{D}}\}$, where $0_{\mathbf{D}} < \mu_D < 1_{\mathbf{D}}$.

*Remark* 4.2. $\mathbf{D}$ has two nontrivial subalgebras:

$$\mathbf{2} := (\{1, 0\}, \cdot, d, 1), \text{ where } d \equiv 1,$$

$$\mathbf{2}^{\wedge} := (\{1, *\}, \cdot, d, 1), \text{ where } d(x, y) := x \wedge y.$$

Thus, the algebra $\mathbf{2}$ is a Boolean group and is abelian, while the algebra $\mathbf{2}^{\wedge}$ is a Boolean algebra without zero [18] and is not abelian. Note that $\mathbf{D}/\mu_{\mathbf{D}} \cong \mathbf{2}$, and, consequently, $\mathbf{A} \in HS(\mathbf{D})$ iff $\mathbf{A} \cong \mathbf{2}$ or $\mathbf{A} \cong \mathbf{2}^{\wedge}$ or $\mathbf{A} \cong \mathbf{D}$ for non-trivial $\mathbf{A} \in \mathcal{V}(\mathbf{D})$.

Now, applying [6, Theorem 2.10] we get immediately:

Proposition 4.3. $\mathcal{V}(\mathbf{D})$ is a Fregean variety.

Next, we look at the commutator operation in $\operatorname{Con}\mathbf{D}$.

Proposition 4.4.

1. $[\mu_{\mathbf{D}}, \mu_{\mathbf{D}}] = \mu_{\mathbf{D}}$,
2. $[1_{\mathbf{D}}, 1_{\mathbf{D}}] = \mu_{\mathbf{D}}$,
3. $(0_{\mathbf{D}} : \mu_{\mathbf{D}}) = 0_{\mathbf{D}}$,
4. $(\mu_{\mathbf{D}} : 1_{\mathbf{D}}) = 1_{\mathbf{D}}$.

Proof: (1) From the definition of the commutator we get:

$$1 = d(1,1) \equiv_{[\mu_{\mathbf{D}},\mu_{\mathbf{D}}]} d(1,*) = * \quad \text{iff}$$

$$* = d(*,1) \equiv_{[\mu_{\mathbf{D}},\mu_{\mathbf{D}}]} d(*,*) = *,$$

so $(1,*) \in [\mu_{\mathbf{D}}, \mu_{\mathbf{D}}]$, thus $[\mu_{\mathbf{D}}, \mu_{\mathbf{D}}] = \mu_{\mathbf{D}}$.

(2) Since $\mathbf{D}/\mu_{\mathbf{D}} \cong \mathbf{2}$, we get immediately from the general property of the commutator operation [2]:

$$\mu_{\mathbf{D}}/\mu_{\mathbf{D}} = [1_{\mathbf{D}}/\mu_{\mathbf{D}}, 1_{\mathbf{D}}/\mu_{\mathbf{D}}] = ([1_{\mathbf{D}}, 1_{\mathbf{D}}] \vee \mu_{\mathbf{D}})/\mu_{\mathbf{D}}.$$

Thus $[1_{\mathbf{D}}, 1_{\mathbf{D}}] \vee \mu_{\mathbf{D}} = \mu_{\mathbf{D}}$, and consequently $[1_{\mathbf{D}}, 1_{\mathbf{D}}] \subseteq \mu_{\mathbf{D}}$. From the equality $[\mu_{\mathbf{D}}, \mu_{\mathbf{D}}] = \mu_{\mathbf{D}}$ we get $[1_{\mathbf{D}}, 1_{\mathbf{D}}] \subseteq \mu_{\mathbf{D}} \subseteq [\mu_{\mathbf{D}}, \mu_{\mathbf{D}}]$, and therefore $[1_{\mathbf{D}}, 1_{\mathbf{D}}] = \mu_{\mathbf{D}}$.

(3), (4) From Definition 2.3 and (1) and (2) we get that $1_{\mathbf{D}}$ is Abelian over $\mu_{\mathbf{D}}$, and $\mu_{\mathbf{D}}$ is not Abelian in $\mathbf{D}$. Thus, from [5, Lemma 21] we obtain the assertion. □

From the above proposition we get the following result:

Corollary 4.5. The algebra $\mathbf{D}$ is polynomially equivalent neither to the three-element equivalential algebra nor to the three-element Brouwerian semillatice.

Proposition 4.6. There are only three (up to isomorphism) nontrivial subdirectly irreducible algebras in $\mathcal{V}(\mathbf{D})$: $\mathbf{D}, \mathbf{2}, \mathbf{2}^{\wedge}$.

Proof: From Remark 4.2 we know that up to isomorphism the only nontrivial subdirectly irreducible algebras in $HS(\mathbf{D})$ are: $\mathbf{D}, \mathbf{2}, \mathbf{2}^{\wedge}$. Among them only $\mathbf{2}$ has an abelian monolith. Suppose that $\mathbf{B} := \{B, \cdot, d_B, 1\}$ is subdirectly irreducible in $\mathcal{V}(\mathbf{D})$. It follows from [2, Theorem 10.12] that there exists a subdirectly irreducible algebra $\mathbf{A} \in HS(\mathbf{D})$ such that either $\mathbf{B} \cong \mathbf{A}$ or $\mathbf{B}$ and $\mathbf{A}$ have abelian monoliths and $\mathbf{B}/(0_{\mathbf{B}} : \mu_{\mathbf{B}}) \cong \mathbf{A}/(0_{\mathbf{A}} : \mu_{\mathbf{A}})$. Thus $\mathbf{B} \in \{\mathbf{D}, \mathbf{2}, \mathbf{2}^{\wedge}\}$ (up to isomorphism) or $\mathbf{B}$ has an abelian monolith and $\mathbf{B}/(0_{\mathbf{B}} : \mu_{\mathbf{B}}) \cong \mathbf{2}/(0_{\mathbf{2}} : \mu_{\mathbf{2}})$. Assume that the second possibility holds. From (**SC1**) we get $(0_{\mathbf{B}} : \mu_{\mathbf{B}}) = \mu_{\mathbf{B}}$. Thus $\mathbf{B}/\mu_{\mathbf{B}} \cong \mathbf{2}/(0_{\mathbf{2}} : \mu_{\mathbf{2}})$. Since $\mathbf{2}/(0_{\mathbf{2}} : \mu_{\mathbf{2}}) = \mathbf{2}/\mu_{\mathbf{2}}$ is a trivial algebra, it follows from Proposition 2.7 that $B$ with the natural order is the two-element chain, and so $B = \{1, 0\}$. Using identities $d_B(x,1) \approx d_B(x,x) \approx d_B(1,x)$ and $d_B(1,1) \approx 1$, true in $\mathcal{V}(\mathbf{D})$, we get $d_B(1,0) = d_B(0,1) = d_B(0,0)$. Suppose that $d_B(1,0) = 0$, then $d_B(x,y) = x \wedge y$, contrary to the fact that $\mathbf{B}$ has an abelian monolith. Thus $d_B(1,0) = 1$, and so $d_B \equiv 1$. In consequence $\mathbf{B} \cong \mathbf{2}$, which completes the proof. □

*Remark* 4.7. It follows from Proposition 4.6 that all subdirectly irreducible algebras in $\mathcal{V}(\mathbf{D})$ belong to $S(\mathbf{D})$. Thus $\mathcal{V}(\mathbf{D}) = SP(\mathbf{D})$. In consequence, a quasivariety generated by $\mathbf{D}$ turns out to be a variety.

## 5. Frames for the algebras from $\mathcal{V}(\mathbf{D})$

Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$. Recall, that we denote by $\mathrm{Cm}(\mathbf{A})$ the set of all completely meet-irreducible congruences on $\mathbf{A}$. In this section we describe an additional structure (frame) on $\mathbf{Cm}(\mathbf{A})$. This structure is similar to the frames in the equivalential algebras with conjunction on the regular elements described in [11].

It follows from Proposition 4.6 that $\mu \in \mathrm{Cm}(\mathbf{A})$ iff $\mathbf{A}/\mu \cong \mathbf{k}$, for $\mathbf{k} \in \{\mathbf{D}, \mathbf{2}, \mathbf{2}^{\wedge}\}$. We use the following notation:

$$\overline{L} := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}\},$$

$$\underline{L} := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{D}\},$$

$$P := \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}^{\wedge}\},$$

$$L := \overline{L} \cup \underline{L}.$$

Proposition 5.1. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and $\mu \in \mathrm{Cm}(\mathbf{A})$. If $\mathbf{A}/\mu \cong \mathbf{D}$, then $\mathbf{A}/\mu^{+} \cong \mathbf{2}$.

Proof: Let $f : \mathbf{A}/\mu \to \mathbf{2}$ be the function given by $f(1/\mu) = f(*/\mu) = 1$ and $f(0/\mu) = 0$. Therefore $f$ is a surjective homomorphism and $\ker f = \mu^{+}/\mu$. Thus $(\mathbf{A}/\mu)/(\mu^{+}/\mu) \cong \mathbf{2}$, and consequently $\mathbf{A}/\mu^{+} \cong \mathbf{2}$. □

Corollary 5.2. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and $\mu \in \mathrm{Cm}(\mathbf{A})$. Then $\mu \in P \cup \overline{L}$ iff $\mu \prec \mathbf{1_A}$ (i. e., $\mu^{+} = \mathbf{1_A}$) and $\mu \in \underline{L}$ iff $\mu^{+} \in \overline{L}$.

In consequence, the length of the longest chain in $\mathrm{Cm}(\mathbf{A})$ equals two.

Let $\mathbf{A} \in \mathcal{V}(\mathbf{R})$ and $\varphi, \psi \in \mathrm{Cm}(\mathbf{A})$. We introduce a relation on $\mathrm{Cm}(\mathbf{A})$ as follows (see [5, p. 51]):

$$\varphi \sim \psi \text{ iff the intervals } I[\varphi, \varphi^{+}] \text{ and } I[\psi, \psi^{+}] \text{ are projective.}$$

It is easy to see, that the relation $\sim$ is an equivalence relation on $\mathrm{Cm}(\mathbf{A})$.

From [17, Lemma 4.2, Corollary 3.7] it follows that the definition of the relation $\sim$ is equivalent to the following definition: $\varphi \sim \psi$ iff $\varphi^+ = \psi^+$ and $\varphi \bullet \psi \in \mathrm{Cm}(\mathbf{A})$, where $\varphi \bullet \psi = (\varphi \div \psi)' \cap \varphi^+$.

Definition 5.3. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$. The structure $\mathbf{Cm}(\mathbf{A}) := (\mathrm{Cm}(\mathbf{A}), \leq, \sim)$ is called a **frame** of $\mathbf{A}$, where $\leq$ is the inclusion relation.

First, we show that the relation $\sim$ on $P \cup \underline{L}$ is an identity.

Proposition 5.4. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and $\mu \in P \cup \underline{L}$. Then $|\mu/_\sim| = 1$.

Proof: Let $\mu \in P$. Then $\mu^+ = 1_\mathbf{A}$. Since $\mathbf{A}/\mu$ is not Abelian, so from [2, Proposition 3.7] we get that $1_\mathbf{A}$ is not Abelian over $\mu$. Thus $\mu^+$ is not Abelian over $\mu$. Let now $\mu \in \underline{L}$. Then $\mathbf{A}/\mu \cong \mathbf{D}$. Since $\mu_\mathbf{D}$ is not Abelian, thus $\mu^+/\mu$, the monolith of $\mathbf{A}/\mu$, is also not Abelian. In both cases, from [5, Lemma 21] we have $\mu/_\sim = \{\mu\}$. □

Theorem 5.5. *Let* $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ *and* $\mu \in \overline{L}$. *Then:*
1) $\mu/_\sim = \{\nu \in L : \nu^+ = 1_\mathbf{A}\} = \overline{L}$,
2) $(\mu/_\sim \cup \{1_\mathbf{A}\}, \bullet)$ *forms a Boolean group, where* $\mu_1 \bullet \mu_2 := (\mu_1 \div \mu_2)'$ *for* $\mu_1, \mu_2 \in \mu/_\sim$.

Proof: (1) From [5, Lemma 21] we know that $\mu/_\sim \subseteq \{\nu \in L : \nu^+ = 1_\mathbf{A}\}$. We need to prove the reverse inclusion. Let $\varphi \in \{\nu \in L : \nu^+ = 1_\mathbf{A}\}$ and $\varphi \neq \mu$. First we show that $\mu \bullet \varphi$ is a congruence on $\mathbf{A}$. Since $\mathrm{Cm}(\mathbf{A}) \subseteq \mathrm{Cm}(\mathbf{A}^e)$, see [7, Lemma 4.1], we have $\mu, \varphi \in \mathrm{Cm}(\mathbf{A}^e)$. Thus, from [14, Proposition 3] we get that $\mu \bullet \varphi \in \mathrm{Con}\,\mathbf{A}^e$. Next, we show that the relation $\mu \bullet \varphi$ is compatible with the operation $d$. Let $(a, b), (e, f) \in \mu \bullet \varphi$. Since the operation $d \equiv 1$ on $\mathbf{A}/\mu \cup \mathbf{A}/\varphi$, we get $d(a, e) \cdot d(b, f) \in 1/\mu$ and $d(a, e) \cdot d(b, f) \in 1/\varphi$. Thus $d(a, e) \cdot d(b, f) \in 1/\mu \wedge \varphi$, and, consequently, $(d(a, e), d(b, f)) \in \mu \wedge \varphi \subseteq \mu \bullet \varphi$. Therefore $\mu \bullet \varphi$ is a congruence. Since $\mu^+ = \varphi^+$, from [17, Corollay 3.7] we get $\mu \sim \varphi$. Thus $\varphi \in \mu/_\sim$, and so $\{\nu \in L : \nu^+ = 1_\mathbf{A}\} \subseteq \mu/_\sim$.

The assertion (2) follows from [17, Theorem 3.6]. □

Summarizing, the equivalence classes of the relation $\sim$ on $\mathrm{Cm}(\mathbf{A})$ take the following form:

1. $\overline{L} \in \mathrm{Cm}(\mathbf{A})/_\sim$,
2. $\mu/_\sim = \{\mu\}$ for all $\mu \in \underline{L} \cup P$.

## 6. Representation theorem

A maximal proper subalgebra of the Boolean group is called a **hyperplane**. We use this word, because a Boolean group can be interpreted as a vector space over the field $\mathbb{Z}_2$. We will write $Z\uparrow := \{\nu \in \mathrm{Cm}(\mathbf{A}) : \nu \geq \mu \text{ for some } \mu \in Z\}$ and $Z\downarrow := \{\nu \in \mathrm{Cm}(\mathbf{A}) : \nu \leq \mu \text{ for some } \mu \in Z\}$ for $Z \subseteq \mathrm{Cm}(\mathbf{A})$. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$. To get the representation theorem, we need to define a family of subsets on the set $\mathrm{Cm}(\mathbf{A})$ called the hereditary sets. This idea came from Słomczyńska, see [14]. The general definition [17, Definition 4.5] works for every algebra $\mathbf{A}$ from a Fregean variety. It is easy to see that in our case this definition takes the following form:

Definition 6.1. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and $Z \subseteq \mathrm{Cm}(\mathbf{A})$. A set $Z$ is **hereditary** if:

1. $Z = Z\uparrow$,
2. $\overline{L} \subseteq Z$ or $((\overline{L} \cap Z) \cup \{\mathbf{1_A}\}, \bullet)$ is a hyperplane in $(\overline{L} \cup \{\mathbf{1_A}\}, \bullet)$.

We denote by $\mathcal{H}(\mathbf{A})$ the set of all hereditary subsets of $\mathrm{Cm}(\mathbf{A})$.

We define a map $M$ as follows:

$$M : A \ni a \to M(a) := \{\mu \in \mathrm{Cm}(\mathbf{A}) : a \in 1/\mu\},$$

for all $\mathbf{A} \in \mathcal{V}(\mathbf{D})$.

Now, we formulate the representation theorem.

Theorem 6.2. *Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and let $\mathbf{A}$ be finite. Then the map $M : A \ni a \to M(a) := \{\mu \in \mathrm{Cm}(\mathbf{A}) : a \in 1/\mu\}$ is the isomorphism between $\mathbf{A}$ and $(\mathcal{H}(\mathbf{A}), \leftrightarrow, \mathsf{d}, \boldsymbol{1})$, where*

$$Z \leftrightarrow Y := ((Z \div Y)\downarrow)'$$

$$\mathsf{d}(Z, Y) := [Z \cup ((Z\downarrow)' \cap L)] \cap [Y \cup ((Y\downarrow)' \cap L)],$$

$$\boldsymbol{1} := \mathrm{Cm}(A),$$

*for $Z, Y \in \mathcal{H}(\mathbf{A})$.*

Proof: From [17, Proposition 4.8] we deduce that $M(a)$ is a hereditary set, so the map $M$ is well defined. Next, we conclude from [17, Theorem 4.14] that $M$ is a bijection which preserves the equivalence operation. Clearly, if $Z = \mathrm{Cm}(\mathbf{A})$, then $Z = M(1)$. Thus, it suffices to show that $M$ preserves $\mathsf{d}$.

Of course, we have $\overline{L} \subseteq \mathsf{d}(Z,Y)$ for all $Z, Y \in \mathcal{H}(\mathbf{A})$, and so $\mathsf{d}(Z,Y)$ is a hereditary set. Moreover, $Z \cap Y \subseteq \mathsf{d}(Z,Y)$. We show that

$$M(d(a,b)) = [M(a) \cup ((M(a)\downarrow)' \cap L)] \cap [M(b) \cup ((M(b)\downarrow)' \cap L)],$$

for all $a, b \in A$. We recall that if $\mu \in \overline{L}$, then $d(a,b)/\mu = 1/\mu$, and if $\mu \in P$, then $d(a,b)/\mu = a/\mu \wedge b/\mu$. We show inclusion both ways.

„$\subseteq$" Let $\mu \in M(d(a,b))$. We need consider three cases:

1) $\mu \in \overline{L}$. Then the inclusion is obvious.

2) $\mu \in P$. Then

$$\mu \in M(d(a,b)) \Rightarrow d(a,b) \in 1/\mu \Rightarrow d(a,b)/\mu = 1/\mu \Rightarrow$$

$$a/\mu = 1/\mu \text{ and } b/\mu = 1/\mu \Rightarrow a \in 1/\mu \text{ and } b \in 1/\mu \Rightarrow$$

$$\mu \in M(a) \text{ and } \mu \in M(b) \Rightarrow \mu \in M(a) \cap M(b).$$

3) $\mu \in \underline{L}$. In this situation we get $\mu \in M(d(a,b)) \Rightarrow d(a,b)/\mu = 1/\mu \Rightarrow a/\mu \neq */\mu$ and $b/\mu \neq */\mu$. The following cases are possible:

a) $a/\mu = b/\mu = 1/\mu$. Then $\mu \in M(a) \cap M(b)$.

b) $a/\mu = b/\mu = 0/\mu$. Therefore

$$a/\mu^+ = b/\mu^+ = 0/\mu^+ \Rightarrow a, b \notin 1/\mu^+ \Rightarrow \mu^+ \notin M(a) \text{ and } \mu^+ \notin M(b) \Rightarrow$$

$$\mu \notin M(a)\downarrow \text{ and } \mu \notin M(b)\downarrow \Rightarrow \mu \in (M(a)\downarrow)' \text{ and } \mu \in (M(b)\downarrow)'.$$

Hence $\mu \in [M(a) \cup ((M(a)\downarrow)' \cap L)] \cap [M(b) \cup ((M(b)\downarrow)' \cap L)]$.

c) $a/\mu = 1/\mu$, $b/\mu = 0/\mu$ (or vice versa). Then $a \in 1/\mu$, so $\mu \in M(a)$. Since $b \notin 1/\mu^+$, so $\mu \notin M(b)\downarrow$, and consequently $\mu \in (M(b)\downarrow)'$. Thus

$$\mu \in [M(a) \cup ((M(a)\downarrow)' \cap L)] \cap [M(b) \cup ((M(b)\downarrow)' \cap L)].$$

„$\supseteq$" Let $\mu \in [M(a) \cup ((M(a)\downarrow)' \cap L)] \cap [M(b) \cup ((M(b)\downarrow)' \cap L)]$. Once again we need consider three cases:

1) $\mu \in \overline{L}$. Then $d(a,b)/\mu = 1/\mu \Rightarrow d(a,b) \in 1/\mu \Rightarrow \mu \in M(d(a,b))$.

2) $\mu \in P$. In this case:

$$\mu \in M(a) \cap M(b) \Rightarrow \mu \in M(a) \text{ and } \mu \in M(b) \Rightarrow a/\mu = 1/\mu$$
$$\text{and } b/\mu = 1/\mu \Rightarrow d(a,b)/\mu = 1/\mu \Rightarrow d(a,b) \in 1/\mu \Rightarrow \mu \in M(d(a,b)).$$

3) $\mu \in \underline{L}$. Let us consider the following cases:

a) $\mu \in M(a)$ and $\mu \in M(b)$. Then

$$a, b \in 1/\mu \Rightarrow a/\mu = b/\mu = 1/\mu \Rightarrow d(a,b)/\mu = 1/\mu \Rightarrow$$
$$d(a,b) \in 1/\mu \Rightarrow \mu \in M(d(a,b)).$$

b) $\mu \in M(a)$ and $\mu \in (M(b)\downarrow)'$ (or analogously: $\mu \in (M(a)\downarrow)'$ and $\mu \in M(b)$). Therefore $a/\mu = 1/\mu$ and $b/\mu = 0/\mu$. Then $d(a,b)/\mu = 1/\mu$, so $d(a,b) \in 1/\mu$, and consequently $\mu \in M(d(a,b))$.

c) $\mu \in (M(a)\downarrow)'$ and $\mu \in (M(b)\downarrow)'$. Then $a/\mu = b/\mu = 0/\mu$, so we get as above $d(a,b)/\mu = 1/\mu$. Thus $d(a,b) \in 1/\mu$, and hence $\mu \in M(d(a,b))$. Finally, we conclude that $M$ preserves $\mathsf{d}$, and so $M$ is the isomorphism as claimed. □

*Example* 6.3. Let $A = \{*, 1\}^3 \cup \{(1,0,0), (0,1,0), (0,0,1), (*,0,0), (0,*,0), (0,0,0)\}$. Thus $A$ is closed under equivalence operation $\cdot$ and $(A, \cdot)$ is the smallest equivalential algebra, which is not a reduct of a Heyting algebra, see [13, Example 3]. Moreover, $d(x,y) \in \{1,*\}^3$ for all $x, y \in \mathbf{D}^3$. Therefore $\mathbf{A} = (A, \cdot, d) \in S(\mathbf{D}^3)$.

Let us consider three subsetes of $A$: $F_1 := \{*,1\}^3 \cup \{(1,0,0), (*,0,0)\}$, $F_2 := \{*,1\}^3 \cup \{(0,1,0), (0,*,0)\}$ and $F_3 := \{*,1\}^3 \cup \{(0,0,1), (0,0,*)\}$. Then the relations $\mu_i$ for $i \in \{1,2,3\}$, defined by: $a \equiv_{\mu_i} b$ iff $ab \in F_i$ for all $a, b \in A$, are congruences of $\mathbf{A}$. Moreover, an easy computation shows that $\mathbf{1}/\mu_i = F_i$ for all $i \in \{1,2,3\}$ (where $\mathbf{1} = (1,1,1)$) and $a/\mu_i = A \setminus F_i$ for all $a \in A \setminus F_i$. Choosing $a = (a_1, a_2, a_3) \in \{0,1\}^3$ for every $i \in \{1,2,3\}$, we get: $d(\mathbf{1}/\mu_i, a/\mu_i) = (d(1,a_1), d(1,a_2), d(1,a_3))/\mu_i = \mathbf{1}/\mu_i$. Therefore $\mathbf{A}/\mu_i \cong \mathbf{2}$.

Next, let us consider 5-element subsets $G_i \subseteq F_i$, for $i \in \{1,2,3\}$: $G_1 := \{(1,x,y) : x, y \in \{1,*\}\} \cup \{(1,0,0)\}$, $G_2 := \{(x,1,y) : x, y \in \{1,*\}\} \cup \{(0,1,0)\}$, $G_3 := \{(x,y,1) : x, y \in \{1,*\}\} \cup \{(0,0,1)\}$. Relations $\nu_i$, which are designated by these subsetes ($a \equiv_{\nu_i} b$ iff $ab \in G_i$), are congruences of $\mathbf{A}$. Moreover, $\mathbf{1}/\nu_i = G_i$ and $c/\nu_i = F_i \setminus G_i$, $a/\nu_i = a/\mu_i$ for all $c \in F_i \setminus G_i$, $a \in A \setminus F_i$. Thus $\mathbf{A}/\nu_i \cong \mathbf{D}$.

Finally, we get that $\mathrm{Cm}(\mathbf{A})$ has the form as shown in Figure 2. It is easy to check that, according to Theorem 6.2, this frame corresponds to the 14-element algebra $\mathbf{A}$. We can also deduce that $\mathbf{A}$ is directly irreducible.

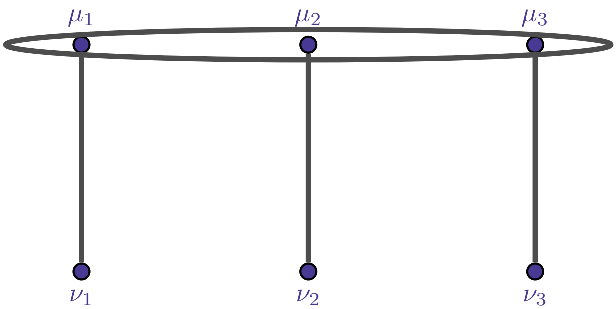


**Figure 2.**

In general situation, we show that every finite algebra from $\mathcal{V}(\mathbf{D})$ can be naturally decomposed as the direct product of two algebras. Recall that $L = \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2} \text{ or } \mathbf{A}/\mu \cong \mathbf{D}\}$ and $P = \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}^\wedge\}$.

Proposition 6.4. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ be finite. Then:

$$\mathbf{A} \cong \mathbf{A}/_{\bigcap L} \times \mathbf{A}/_{\bigcap P}.$$

Proof: As $\mathbf{A}$ is finite, so $\mathbf{1_A} = \bigvee_{i=1}^{n} \alpha_i$, where $\alpha_i$ ($i \in \{1, \ldots, n\}$) are join-irreducible congruences. Clearly, $\bigcap L \wedge \bigcap P = \mathrm{Cm}(\mathbf{A}) = 0_{\mathbf{A}}$. We need to prove that $\alpha_i \subseteq \bigcap L \vee \bigcap P$ for all $i \in \{1, \ldots, n\}$. Let $i \in \{1, \ldots, n\}$. Assume that $\alpha_i \nsubseteq \bigcap L$. We show that $\alpha_i \subseteq \bigcap P$. Suppose, contrary to our claim, that there exists $\mu \in P$ such that $\alpha_i \nsubseteq \mu$. Then $\alpha_i \vee \mu = \mathbf{1_A}$ and $\alpha_i \wedge \mu < \alpha_i$. Thus the intervals $I[\alpha_i \wedge \mu, \alpha]$ and $I[\mu, \mathbf{1_A}]$ are projective, and, consequently, $\alpha_i \wedge \mu = \alpha_i^-$. On the other hand, there exists $\nu \in L$ such that $\alpha_i \nsubseteq \nu$ and $\alpha_i \subseteq \nu^+$. Therefore, the intervals $I[\alpha_i^-, \alpha]$ and $I[\nu, \nu^+]$ are projective. Thus, we get $\nu \sim \mu$, a contradiction. $\square$

# 7. Free algebras in $\mathcal{V}(\mathbf{D})$ – a sketch of construction

Now, we can construct the finitely generated free algebras in $\mathcal{V}(\mathbf{D})$. We will denote by $F_{\mathbf{D}}(n)$ the free $n$-generated algebra in $\mathcal{V}(\mathbf{D})$ in which $X = \{x_1, x_2, \dots, x_n\}$ is the $n$-element set of free generators. Observe that if $\mu \in \mathrm{Cm}(F_{\mathbf{D}}(n))$, then we can identify $\mu$ with a map $f$ which sends free generators in $\mathbf{k}$, where $\mathbf{k} \in \{\mathbf{D}, \mathbf{2}, \mathbf{2}^{\wedge}\}$, in such a way that $f^{-1}(\{*\}) \neq \emptyset$. This map can be uniquely extended to a surjective homomorphism $\overline{f}\colon F_{\mathbf{D}}(n) \longrightarrow \mathbf{k}$. It follows that $\ker \overline{f} \in \mathrm{Cm}(F_{\mathbf{D}}(n))$. So, the construction of the frame $\mathrm{Cm}(F_{\mathbf{D}}(n))$ is similar to the construction of the frame of the equivalential algebras with conjunction on the regular elements, described in [11].

This construction proceeds as follows:

1. Each $\mu \in \mathrm{Cm}(F_{\mathbf{D}}(n))$ is labelled by the set indices $\{i : x_i \in X \cap (1/\mu)\} \subseteq \{1, \dots, n\}$.
2. $\overline{L}$ has $2^n - 1$ elements labelled by all proper subsets of $\{1, \dots, n\}$ and these elements form only one equivalence class.
3. $P$ has $2^n - 1$ elements also labelled by all proper subsets of $\{1, \dots, n\}$, but in this case each element forms a one-element equivalence class.
4. If $\mu \in \overline{L}$ is labelled by $S \subsetneq \{1, \dots, n\}$, then below $\mu$ (i. e., in $\underline{L}$) there are elements labelled by all proper subsets of $S$.
5. Each $\mu \in \underline{L}$ forms a one-element equivalence class.

In the figures below:

*a.* Each dot denotes an element of the frame.

*b.* Straight lines denote a partial ordering directed upwards.

*c.* The equivalence class with more than one element is marked with an ellipse.

*d.* Each dot that does not lie in an ellipse denotes a one-element equivalence class.

## 7.1. The frame of $F_{\mathbf{D}}(2)$ – the free algebra in $\mathcal{V}(\mathbf{D})$ with two free generators

The set $\mathrm{Cm}(F_{\mathbf{D}}(2))$ has 8 elements (Figure 3): 5 on the left-hand side (all elements at the top form one equivalence class and the elements at the

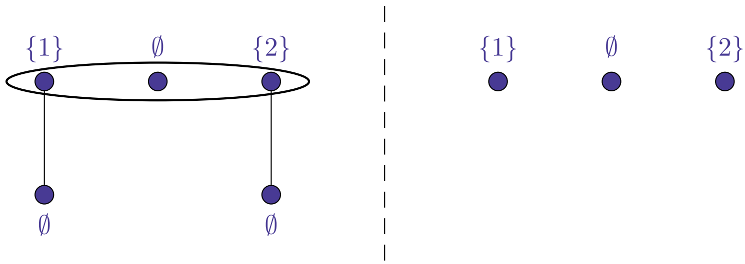


**Figure 3.** $\mathbf{Cm}(F_{\mathbf{D}}(2))$

bottom form one-element equivalence classes) and 3 on the right-hand side (each in a separate equivalence class). So, there are 9 hereditary sets on the left-hand side and 8 hereditary sets on the right-hand side. Finally, $|F_{\mathbf{D}}(2)| = 9 \cdot 8 = 72$.

## 7.2. The frame of $F_{\mathbf{D}}(3)$ – the free algebra in $\mathcal{V}(\mathbf{D})$ with three free generators

The set $\mathrm{Cm}(F_{\mathbf{D}}(3))$ has 26 elements (Figure 4): 7 on the left-hand side at the top, 12 on the left-hand side at the bottom, and 7 on the right-hand side. On the left-hand side there are 4536 hereditary sets, and on the right-hand side there are 128 hereditary sets. Finally, $|F_{\mathbf{D}}(3)| = 4536 \cdot 128 = 580608$.

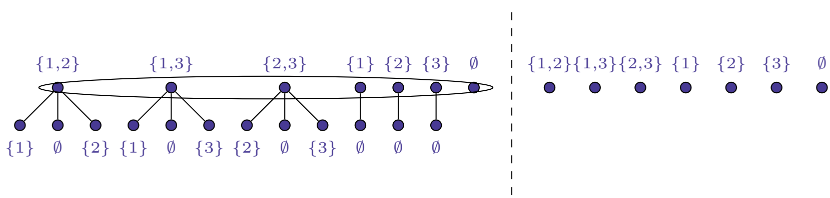


**Figure 4.** $\mathbf{Cm}(F_{\mathbf{D}}(3))$

Using Theorem 6.2 and the construction above one can also find the formula for the free spectrum. We plan to publish these result in the next article, which will be a continuation of this paper.

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