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ON PARACOMPLETE VERSIONS OF JAŚKOWSKI'S DISCUSSIVE LOGIC

Abstract

Jaśkowski's discussive (discursive) logic $\mathbf{D_2}$ is historically one of the first paraconsistent logics, i.e., logics that 'tolerate' contradictions. Following Jaśkowski's idea to define his discussive logic by means of the modal logic **S5** *via* special translation functions between discussive and modal languages and supporting at the same time the tradition of paracomplete logics being the counterpart of paraconsistent ones, we present a paracomplete discussive logic $\mathbf{D_2^{\mathsf{p}}}$.

Keywords: discussive logic, discursive logic, modal logic, paracomplete logic, paraconsistent logic.

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1. Introduction

The idea to set up *paracomplete* versions of Jaśkowski's discussive[1] logic $\mathbf{D_2}$ [23, 22], commonly accepted to be among the pioneering *paraconsis-*

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[1] *Discursive* is another dubbing for this logic. Without any additional reason we choose the former dubbing through the paper.

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tent logics, may appear to the reader as a contradiction in terms.[2] S/He might think that the present paper is, at best, of insignificant and technical importance or is, at worst, about an unconventional and deliberately provocative interpretation of $\mathbf{D_2}$ which has now become a classic. What we would like to avoid in this paper, foremost, is giving the reader the impression that $\mathbf{D_2}$ might be argued not to be paraconsistent. We would not like the reader to get the impression that setting up a system, where $p \rightarrow_{\mathrm{d}} (\neg p \rightarrow_{\mathrm{d}} q)$ is a theorem, is in line with Jaśkowski's original ideas on discussive implication $\rightarrow_{\mathrm{d}}$.

On the contrary, we would like to stress that while presenting his view on the Duns Scotus law (see [23]), Jaśkowski points out that, since antiquity, Aristotle's view that two contradictory statements are not both true has been a subject of criticism. Jaśkowski emphasizes that in the nineteenth and twentieth centuries, these views revived, under which it was pointed out that there are convincing arguments that lead to contradictory conclusions. So, he aimed to construct a system in which the implicational law of overfilling $p \rightarrow (\neg p \rightarrow q)$ is not valid. The idea behind the construction of such a system was as follows: first, with regard to inconsistent sets of statements, such a system does not always lead to overfilling of the set of conclusions; second, it is supposed to be so rich as to enable practical inference; and third, it should have an intuitive justification.

Due to the form of the posed problem, the choice of the implication plays a crucial role in building such a system. Originally, using the modal operator of possibility, Jaśkowski introduced discussive implication and, on its basis, also the discussive equivalence. The system of modal sentences that results from enriching the modal logic $\mathbf{S5}$ with the relevant definitions of discussive connectives is denoted as $\mathbf{M_2}$. On the basis of $\mathbf{M_2}$ Jaśkowski is defining *the system of the two-valued discussive sentential calculus*. This logic is quite rich and allows for the rejection of the implicational law of overfilling.

[2] Following Jaśkowski's approach [23, 22], we call a logic $\mathbf{L}$ paraconsistent iff there are formulas A and B such that $A \rightarrow (\neg A \rightarrow B)$ is not valid in $\mathbf{L}$. Following Akama and da Costa [3], we call a logic $\mathbf{L}$ paracomplete iff there is a formula A such that $A \vee \neg A$ is not valid in $\mathbf{L}$. This definition has already been used by Sette and Carnieli [47], although they preferred the term (nowadays, rarely used) weakly-intuitionistic logic. For the other definitions of paralogics (logics that are paraconsistent or paracomplete), the reader is addressed to [3, 40].

Interestingly, due to the use of classical conjunction in Jaśkowski's first paper [23] on discussive logic, one might suppose that discussive logic is non-adjunctive. And indeed, while presenting examples of formulas that are not theses of discussive logic, Jaśkowski considers $p \rightarrow_{\mathrm{d}} (q \rightarrow_{\mathrm{d}} (p \wedge q))$. He gives intuitions accompanying this: due to the fact that a certain thesis $\mathfrak{P}$ and another thesis $\mathfrak{Q}$ were put forward in the discussion, it does not follow that the thesis $\mathfrak{P} \wedge \mathfrak{Q}$ was also put forward, as it may happen that theses $\mathfrak{P}$ and $\mathfrak{Q}$ were sustained by different participants in the discussion. The intuitive explanation goes along with the formal justification. Of course, the use of classical conjunction leads to $(p \wedge \neg p) \rightarrow_{\mathrm{d}} q$, a version of the overfilling law that is a thesis of such a variant of the discussive system.[3] And only thanks to the non-adjunctive character of it, the implicational version of the overfilling law does not become a thesis.

However, the history of discussive logic does not end with [23]. What is nowadays treated as proper discussive logic is its variant with discussive conjunction. Discussive conjunction completing the language of $\mathbf{D_2}$ is given in a short paper [22]. Only there discussive conjunction $\wedge_{\mathrm{d}}$ is introduced and in this way, the formula $p \rightarrow_{\mathrm{d}} (q \rightarrow_{\mathrm{d}} (p \wedge_{\mathrm{d}} q))$ is becoming a thesis of $\mathbf{D_2}$. On the other hand, $(p \wedge_{\mathrm{d}} \neg p) \rightarrow_{\mathrm{d}} q$ is not a thesis of this final version of $\mathbf{D_2}$. At least to some point, the second paper on discussive logic was much less known than the first one. This was due to the fact that the paper on discussive conjunction was written in Polish and much later translated into English.[4]

On the other hand, we do not mean here Jaśkowski's discussive logic could not tolerate non-standard approaches. See, for example, [32], where an extended version of Jaśkowski's model of discussion with debaters employing modal operators explicitly is presented.

The motivation for this paper stems from conventional sources. First, it correlates to the fact that paraconsistent and paracomplete logics co-exist harmonically, with the latter being a junior counterpart of the former. (See also endnote 21.) Moreover, we argue below that the logic we present here is not the first paracomplete discussive logic in the literature. The second conventional source is to employ modal logic (generally, **S5**) in defining

[3] Already, this fact could be used as a factor showing that Jaśkowski model of discussion can be applied to explore a non-paraconsistent domain as well.

[4] A detailed discussion of the subject may be found in [13].

discussive connectives. Hence, we explore the possibility of changing some of the usual definitions to achieve the clear-cut effect that paracomplete logic results in.[5] Let us detail these sources.

Despite the fact that in [2] Akama, Abe, and Nakamatsu do not dub their *constructive discursive logic with strong negation*, **CDLSN**, paracomplete, there is no counterargument for not doing it. In Section 6 below, we provide a comparative analysis of **CDLSN** and our logic. Here we would like to stress the fact that the Jaśkowskian ideas, which led to one of the first paraconsistent logics ever, permit a discussive paracomplete logic such as **CDLSN** and a non-discussive paracomplete logic such as ours. This view reveals the pair of paraconsistency–paracompleteness as a harmonious tandem rather than a strictly opposed dichotomy. Indeed, the dichotomy in question reveals itself strikingly in the case of many-valued logics.[6] In contradistinction to paraconsistent logic, which is sometimes dubbed *logic with truth-value gluts* because a formula might be true and false simultaneously, paracomplete logic[7] is sometimes dubbed *logic with truth-value gaps* because a formula might be neither true nor false simultaneously. Let us confine ourselves to the case of three-valued logics for this approach to paracomplete logic seems to be the most popular in the literature: A formula of this kind is assigned the third truth-value which is not a designated one. Hence, the law of excluded middle and certain inference rules related to it fail (the *italics* are not ours): "A *paracomplete logic* is a logic, in which the *principle of excluded middle*, i.e., $A \vee \neg A$ is not a theorem of that

[5] Notice that the first formal system, consciously conceived as a logic invalidating Duns Scotus law, was developed by Stanisław Jaśkowski in 1948 [23], while ideas that can be regarded as the basis of paracomplete logics were explored in the 1960s (for example, [50]), with formal investigations in [29].

[6] Note that each logic that is thoroughly discussed in this paper is bivalent. Many-valuedness is needed to clarify the argument. For this aim, we use the terms 'logic with truth-value gluts' and 'logic with truth-value gaps', which come from many-valued logic. At that, we warn against the identification of 'logic with truth-value gluts' with paraconsistent logic and 'logic with truth-value gaps' with paracomplete logic. First, not all paraconsistent and paracomplete logics are many-valued. Second, logics with truth-value gluts and gaps are not always paraconsistent and paracomplete ones. As was shown in [48], many-valued logics (with gaps and gluts) satisfying Rosser and Turquette's standard conditions [45, p. 26] have classical consequence relation. However, in many-valued semantics, gluts usually lead to paraconsistency, and gaps usually lead to paracompleteness.

[7] Among the first papers where this term appears are [29, 30].

logic" [3, p. 8]. Note that $A \to (\neg A \to B)$, one of the Jaśkowskian stimuli, might be a paracomplete theorem, and the situation is upside-down in the case of paraconsistency ($A \vee \neg A$ is a $\mathbf{D_2}$-theorem).

To conclude the detalization of the second source, we find it proper to briefly enlist some of the most famous paracomplete logics (apart from intuitionistic logic and Kolmogorov's system [25]): Kleene's strong three-valued logic $\mathbf{K_3}$ [24], whose absence of theoremhood led to setting up three-valued paracomplete logics with theoremhoods and robust implications (the ones validating modus ponens, etc.). Obviously, its implication extensions are paracomplete, among them Łukasiewicz's three-valued logic $\mathbf{Ł_3}$ [31] and the three-valued logic **PComp** (Słupecki, Bryll, and Prucnal are likely to be its authors [49]; the author of the name **PComp** is Popov [42]). There are other well-known three-valued paracomplete logics: Kleene's weak three-valued logic $\mathbf{K_3^w}$ [24], Bochvar's $\mathbf{B_3}$ [10], Heyting-Gödel-Jaśkowski's $\mathbf{G_3}$ [20, 18, 21], and Sette and Carnielli's $\mathbf{I^1}$ [47]. As for four-valued paracomplete logics, Pietz and Riveccio's [41] **ETL** deserves attention. Most fuzzy logics are paracomplete (and almost none of them is paraconsistent; see [7, 15] for some rare examples of paraconsistent fuzzy logics). At last, let us mention *paradefinite* [6] or *paranormal* [9, 43] logics, i.e., logics which are both paracomplete and paraconsistent. The most influential logic among them is Anderson-Belnap's **FDE** [5].

The harmonious tandem discussed above seems to be more striking if the reader pays attention to the fact that the Jaśkowskian ideas in question were not axiomatized by himself but later. (Let us, again, refer the reader to *status quo* in [38].) As a result, Jaśkowski's followers give his semi-formal intuitions about connecting robust discussive reasoning with **S5**[8] different (even mutually contradicting) formal insights. Roughly, the key points of those techniques are the same, though: two translation functions from a discussive language into a modal language, and *vice versa* together with the notion of the 'M-counterpart of **S5**' introduced by Perzanowski [39], where 'M' stands for the modal possibility operator. Our approach, in fact, follows the *spirit* of the techniques in question in spite of mirror-like transformations of their key points. The notion of the 'L-counterpart of **S5**', where 'L' stands for the modal necessity operator, is employed because the transformed translation functions employ the necessity operator rather

[8] Furmanowski proves that **S4** is enough to establish this connection [16]. This result was strengthened by Perzanowski [39], and Nasieniewski and Pietruszczak [33, 34].

than the possibility one. As one can see, it is not essential for **S5**, but it would be if the whole strategy were applied, for example, to non-normal modal logics. The notion of 'L-counterpart of **S5**' was also introduced by Perzanowski, and it naturally corresponds to the set **LS5** of all those **S5**-theses at the beginning of which there is the necessity operator that was used, in particular, by Kotas in giving his axiomatization of $\mathbf{D_2}$. Now the ray highlights discussive left-, right disjunction rather than discussive left-, right conjunction, as it has been done in the literature before. Hence, the difference between paraconsistency and paracompleteness in the discussive setting is shifted from the standard criterion that certain formulae are (not) theorems of the logic in question to the non-standard criterion that certain connectives are to be especially treated within certain discussive logics. On this path, we believe, some alternatives to Jaśkowski's, Akama-Abe-Nakamatsu's, and our approaches to discussive logic might be discovered.

2. On Jaśkowski's discussive logic $\mathbf{D_2}$

Following Omori and Alama [38], we distinguish between the three languages, $\mathscr{L}$, $\mathscr{L}_\mathrm{r}$, and $\mathscr{L}_\mathrm{l}$, over which $\mathbf{D_2}$ can be built. Note that such a possibility is also used in a question asked by João Marcos and considered in [35]. The former has the alphabet $\{\mathcal{P}, \neg, \vee, \rightarrow_\mathrm{d}, \wedge, (,)\}$, where $\mathcal{P} = \{p, q, r, s, p_1, \ldots\}$ is the set of propositional variables. The languages $\mathscr{L}_\mathrm{r}$ and $\mathscr{L}_\mathrm{l}$ have right and left discussive conjunctions $\wedge^\mathrm{r}_\mathrm{d}$ and $\wedge^\mathrm{l}_\mathrm{d}$, respectively, instead of $\wedge$. The sets of all $\mathscr{L}$-, $\mathscr{L}_\mathrm{r}$-, and $\mathscr{L}_\mathrm{l}$-formulas are defined in the standard way and denoted *via* $\mathscr{F}$, $\mathscr{F}_\mathrm{r}$, and $\mathscr{F}_\mathrm{l}$, respectively. We denote a propositional variable (in the metalanguage) by P, Q, etc., a discussive formula by A, B, C, etc., a modal formula by φ, ψ, γ, etc., and a set of discussive formulas by X. The language $\mathscr{L}_\mathrm{m}$ of the modal logic **S5** has the alphabet $\langle \mathcal{P}, \{\neg, \vee, \rightarrow, \wedge, \Box, \Diamond, (,)\}\rangle$. The set of all $\mathscr{L}_\mathrm{m}$-formulas is defined in the standard way and denoted *via* $\mathscr{F}_\mathrm{m}$. We write $\varphi \leftrightarrow \psi$ for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Following Jaśkowski, we give a translation function τ from $\mathscr{F} \cup \mathscr{F}_\mathrm{r} \cup \mathscr{F}_\mathrm{l}$ into $\mathscr{L}_\mathrm{m}$.

- $\tau(P) = P$, for any $P \in \mathcal{P}$,
- $\tau(\neg A) = \neg\tau(A)$,
- $\tau(A \vee B) = \tau(A) \vee \tau(B)$,
- $\tau(A \rightarrow_\mathrm{d} B) = \Diamond\tau(A) \rightarrow \tau(B)$,
- $\tau(A \wedge B) = \tau(A) \wedge \tau(B)$,
- $\tau(A \wedge^\mathrm{r}_\mathrm{d} B) = \tau(A) \wedge \Diamond\tau(B)$,
- $\tau(A \wedge^\mathrm{l}_\mathrm{d} B) = \Diamond\tau(A) \wedge \tau(B)$.

Jaśkowski [23] originally formulated $\mathbf{D_2}$ in $\mathscr{L}$. However, in his next paper [22] he switched to $\mathscr{L}_\mathrm{r}$.[9] The language $\mathscr{L}_\mathrm{l}$ was used by da Costa, Dubikajtis, Kotas, Achtelik and others [27, 14, 1] as well as by Vasyukov [51].

As noted in [38, Proposition 1], $\tau(A \wedge^\mathrm{r}_\mathrm{d} B) \leftrightarrow \tau(\neg(B \rightarrow_\mathrm{d} \neg A)) \in$ **S5** and $\tau(A \wedge^\mathrm{l}_\mathrm{d} B) \leftrightarrow \tau(\neg(A \rightarrow_\mathrm{d} \neg B)) \in$ **S5**. Obviously, in all three languages $\wedge$ can be expressed *via* $\neg$ and $\vee$. One may think about the fourth conjunction, $\wedge^\mathrm{b}_\mathrm{d}$ (we write 'b' for 'both', $\tau(A \wedge^\mathrm{b}_\mathrm{d} B) \leftrightarrow \Diamond\tau(A) \wedge \Diamond\tau(B))$. However, as noted by Ciuciura [12, p. 85], $(A \wedge^\mathrm{b}_\mathrm{d} B) \rightarrow_\mathrm{d} (\neg(A \wedge^\mathrm{b}_\mathrm{d} B) \rightarrow_\mathrm{d} C)$ will be valid then.

We will denote the formulations of $\mathbf{D_2}$ in $\mathscr{L}_\mathrm{r}$ and $\mathscr{L}_\mathrm{l}$, respectively, *via* $\mathbf{D_2}_\mathrm{r}$ and $\mathbf{D_2}_\mathrm{l}$. The set $\mathbf{D_2}$-tautologies is $\{A \in \mathscr{F} \mid \Diamond\tau(A) \in \mathbf{S5}\}$. Similarly, the sets of $\mathbf{D_2}_\mathrm{r}$- and $\mathbf{D_2}_\mathrm{l}$-tautologies are $\{A \in \mathscr{F}_\mathrm{r} \mid \Diamond\tau(A) \in \mathbf{S5}\}$ and $\{A \in \mathscr{F}_\mathrm{l} \mid \Diamond\tau(A) \in \mathbf{S5}\}$, respectively. Nowadays, $\mathbf{D_2}$ is usually referred to as the corrected version from [22] rather than from [23]: $\mathbf{D_2} = \{\mathbf{A} \in \mathscr{F}_\mathrm{r} \mid \Diamond\tau(\mathbf{A}) \in \mathbf{S5}\}$. It is this version of $\mathbf{D_2}$ with right discussive conjunction that we employ in this paper.

3. Paracomplete versions of $\mathbf{D_2}$

We fix three languages, $\mathscr{L}^*_\mathrm{r}$, $\mathscr{L}^*_\mathrm{l}$, and $\mathscr{L}^*_\mathrm{b}$, over which a variant of $\mathbf{D_2}$ can be built. The former has the alphabet $\{\mathcal{P}, \neg, \vee^\mathrm{r}_\mathrm{d}, \rightarrow^\mathrm{w}_\mathrm{d}, \wedge, (,)\}$, where $\vee^\mathrm{r}_\mathrm{d}$ is right discussive *disjunction*. The language $\mathscr{L}^*_\mathrm{l}$ has left discussive *disjunction* $\vee^\mathrm{l}_\mathrm{d}$ instead of $\vee^\mathrm{r}_\mathrm{d}$.[10] The language $\mathscr{L}^*_\mathrm{b}$ has $\vee^\mathrm{l}_\mathrm{b}$[11] instead of $\vee^\mathrm{r}_\mathrm{d}$. The sets of all $\mathscr{L}^*_\mathrm{r}$-, $\mathscr{L}^*_\mathrm{l}$-, and $\mathscr{L}^*_\mathrm{b}$-formulas are defined in the standard way and denoted *via* $\mathscr{F}^*_\mathrm{r}$, $\mathscr{F}^*_\mathrm{l}$, and $\mathscr{F}^*_\mathrm{b}$, respectively. We define a translation function σ from one of the languages $\mathscr{L}^*_\mathrm{r}$ or $\mathscr{L}^*_\mathrm{l}$ or $\mathscr{L}^*_\mathrm{b}$ to $\mathscr{L}_\mathrm{m}$:

- $\sigma(P) = P$, for any $P \in \mathcal{P}$,
- $\sigma(\neg A) = \neg\sigma(A)$,
- $\sigma(A \wedge B) = \sigma(A) \wedge \sigma(B)$,

[9] Strictly speaking, it was not explicitly said whether the new conjunction was meant to extend the original language or to replace the classical conjunction.

[10] As in the case of Jaśkowski's original model, we also refer to a model of discussion and try to articulate the respective translations in terms of the possible strategy of debaters that could be applied by them while formulating their own statements. That is why we let ourselves to treat the connectives of implication and disjunction as discussive.

[11] Again, 'b' stands for 'both'.

- $\sigma(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} B) = \Box\sigma(A) \rightarrow \sigma(B)$,
- $\sigma(A \vee^{\mathrm{r}}_{\mathrm{d}} B) = \sigma(A) \vee \Box\sigma(B)$,
- $\sigma(A \vee^{\mathrm{l}}_{\mathrm{d}} B) = \Box\sigma(A) \vee \sigma(B)$,
- $\sigma(A \vee^{b}_{d} B) = \Box\sigma(A) \vee \Box\sigma(B)$.

We define the following logics (we associate a logic with its set of tautologies), where $i \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$:

- $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}i} = \{A \in \mathscr{F}^{*}_{i} \mid \Box\sigma(A) \in \mathbf{S5}\}$.

In what follows, we write $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$ for $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}\mathrm{l}}$.

Let us emphasize that, as follows from this definition, $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$ is embeddable into $\mathbf{S5}$ the translation σ. It is not excluded that one can map $\mathbf{S5}$ in $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$, but this issue requires further research.

Let us say a few words about the intuitions that led to such an understanding of discussive connectives. In the original formulation, we have: "If anyone states that p, then q", so from the point of view of a given participant, there is not too much needed to say q. Here we have a much more mistrustful or misgiving position: to say q, it is needed that all participants state p.

However, the reader is not supposed to consider that such a debating model is unrealistic. In a sense, each debating model with some debaters having the power of veto is of this kind. To put it differently, participants are equal in the Jaśkowskian paraconsistent debating model and are not in the paracomplete one in quite the same way debaters are equal in expressing their views on international policy in Łazienki Królewskie, but in order to reach consensus in the United Nations Security Council, the power of veto that its five permanent members have is to be overcome. Generally, to some extent, at least some part of scientific knowledge and juridical process is built in this way: it is only when all the sources (witnesses, observers, experiments, participants in an experiment) jointly say some thesis that a specific conclusion can be added to the current state of knowledge.

One can similarly understand the case of disjunction (say, $\vee^{\mathrm{r}}_{\mathrm{d}}$). Either I am saying p or everyone is stating q, so this would express a kind of dilemma where we have an opposition of my own statements against statements expressed by all the other debaters. Note that the Jaśkowskian disjunction is classical; hence, $p \vee \neg p$ is valid there to the effect that "Everyone is stating that $p \vee \neg p$" expresses the capacity of any participant to state this particular logical law in the course of a debate independently of statements expressed by the other debaters. Paracomplete disjunction here is not classical; hence, $p \vee^{\mathrm{r}}_{\mathrm{d}} \neg p$ is invalid here to the effect that the underlying dilemma "Either I am saying p or everyone is stating that $\neg p$" is, obviously, not characteristic of any debate. To be sure, the paracomplete model of discussion allows any participant to state logical laws, too, but $p \vee^{\mathrm{r}}_{\mathrm{d}} \neg p$ is not among them.

Interestingly, discussive conjunction and disjunction are, in a sense, interdefinable. More strictly, if we would like to compare standard discussive logics and paracomplete discussive ones, we can use some additional translations that show that particular logics are interdefinable. This could be proved inductively by restricting languages to the $\{\neg, \wedge^{\mathrm{l}}_{\mathrm{d}}\}$-part on the discussive side and to the $\{\neg, \vee^{\mathrm{l}}_{\mathrm{d}}\}$-part on the paracomplete side. Of course, this could also be extended to the full languages. So, using inductive hypotheses, we would obtain:

$\tau(\neg(\neg A \wedge^{\mathrm{l}}_{\mathrm{d}} \neg B)) = \neg(\Diamond\neg\tau(A) \wedge \neg\tau(B)) \leftrightarrow \neg\Diamond\neg\tau(A) \vee \neg\neg\tau(B) \leftrightarrow \Box\tau(A) \vee \tau(B) \leftrightarrow_{\textsf{by ind}} \Box\sigma(A) \vee \sigma(B) = \sigma(A \vee^{\mathrm{l}}_{\mathrm{d}} B)$,

$\sigma(\neg(\neg A \vee^{\mathrm{l}}_{\mathrm{d}} \neg B)) = \neg(\Box\neg\sigma(A) \vee \neg\sigma(B)) \leftrightarrow \neg\Box\neg\sigma(A) \wedge \neg\neg\sigma(B) \leftrightarrow \Diamond\sigma(A) \wedge \sigma(B) \leftrightarrow_{\textsf{by ind}} \Diamond\tau(A) \wedge \tau(B) = \tau(A \wedge^{\mathrm{l}}_{\mathrm{d}} B)$.

The logic $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$ has the following axioms (where $\perp$ denotes $p \wedge \neg p$).

Ax$_1$ $A \rightarrow^{\mathrm{w}}_{\mathrm{d}} (B \rightarrow^{\mathrm{w}}_{\mathrm{d}} A)$

Ax$_2$ $(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} (B \rightarrow^{\mathrm{w}}_{\mathrm{d}} C)) \rightarrow^{\mathrm{w}}_{\mathrm{d}} ((A \rightarrow^{\mathrm{w}}_{\mathrm{d}} B) \rightarrow^{\mathrm{w}}_{\mathrm{d}} (A \rightarrow^{\mathrm{w}}_{\mathrm{d}} C))$

Ax$_3$ $\neg\big(A \wedge \neg(A \wedge A)\big)$

Ax$_4$ $\neg\big((A \wedge B) \wedge \neg A\big)$

Ax$_5$ $\neg\big(\neg(A \wedge \neg B) \wedge \neg\neg(\neg(B \wedge C) \wedge \neg\neg(C \wedge A))\big)$

Ax$_6$ $\neg\neg A \rightarrow^{\mathrm{w}}_{\mathrm{d}} A$

Ax$_7$ $\neg(\neg(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \perp) \wedge \neg A)$

Ax$_8$ $\neg(\neg\neg((A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \perp) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \perp) \wedge \neg A)$

Ax$_9$ $\neg\big(\neg(\neg(A \wedge \neg B) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \perp) \wedge \neg\neg(\neg(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \perp) \wedge \neg\neg(B \rightarrow^{\mathrm{w}}_{\mathrm{d}} \perp))\big)$

Ax$_{10}$ $\neg\big(A \wedge \neg\neg(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \perp)\big)$

Ax$_{11}$ $\neg\big(\neg\big(\neg(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \perp) \wedge \neg B\big) \wedge \neg(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} B)\big)$.

Ax$_{12}$ $\neg\big((A \rightarrow^{\mathrm{w}}_{\mathrm{d}} B) \wedge \neg\neg(\neg(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \perp) \wedge \neg B)\big)$

Ax$_{13}$ $\neg\big((A \vee^{\mathrm{l}}_{\mathrm{d}} B) \wedge \neg\neg(\neg\neg(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg B)\big)$

Ax$_{14}$ $\neg\big(\neg(\neg\neg(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg B) \wedge \neg(A \vee^{\mathrm{l}}_{\mathrm{d}} B)\big)$

Ax$_{15}$ $A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \big((\neg\neg\big(\neg(A \wedge \neg B) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot\big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} B\big)$

Ax$_{16}$ $A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \neg(A \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)$

Ax$_{17}$ $\neg(A \wedge \neg B) \rightarrow^{\mathrm{w}}_{\mathrm{d}} (A \rightarrow^{\mathrm{w}}_{\mathrm{d}} B)$

$$\frac{A \quad A \rightarrow^{\mathrm{w}}_{\mathrm{d}} B}{B} \tag{MP$^{\mathrm{w}}_{\mathrm{d}}$}$$

Lemma 3.1 (Deduction theorem). *$X, A \vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} B$ iff $X \vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} A \rightarrow^{\mathrm{w}}_{\mathrm{d}} B$.*

Proof: The proof is textbookian in the presence of Ax$_1$, Ax$_2$, and (MP$^{\mathrm{w}}_{\mathrm{d}}$). □

One can see that any proof given on the basis of classical logic expressed in the language with $\neg$ and $\wedge$ by means of Ax$_3$–Ax$_5$ and the respective form of modus ponens can be transferred into a $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$-proof by Ax$_{17}$.

Fact 3.2. For any thesis A of classical logic in the language with $\neg$ and $\wedge$, $\vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} A$.

3.1. Lewis's intensional implication and disjunction

Intensional implication and disjunction introduced by Lewis in the systems **S1-S5** had a deep influence on modern modal logic (especially the former, which is mostly dubbed *strict implication*). It is the well-knownness of intensional implication and disjunction that allows us to skip details (Lewis's motivation to introduce it, analyzing its pros and cons, etc.) and address those properties of them that concern the purpose of our study only.[12] We begin with strict implication, which we do not denote with the Lewisian fishhook but with $\rightarrow_L$, so that its traditional definition looks as follows: $\varphi \rightarrow_L \psi =_{\mathrm{df}} \Box(\varphi \rightarrow \psi)$. We are interested in two arguments: the one by Jaśkowski who rejects $\rightarrow_L$ in the quality of discussive implication, and the one by Lewis, who rejects the known classical equivalence between implication and disjunction.

[12] An accurate introduction to Lewis's ideas and their impact on modern modal logic is in [8].

Let us remind the reader that Jaśkowski devotes a passage to the Lewisian implication while describing the known solutions to the problem of formulating the logic of inconsistent systems [23, p. 40]. In more detail, he rejects it due to its weakness: "But the set of the theses which include strict implication only, and do not include material implication, is very limited" [23, p. 40].[13] Our implication is stronger than $\to_L$ because $\Box(\varphi \to \psi) \models_{\mathbf{S5}} \Box(\Box\varphi \to \psi)$, and the opposite is false, though.[14] Moreover, in contrast to $\to_L$, our implication is not paraconsistent because $\neg A \to^{\mathrm{w}}_{\mathrm{d}} (A \to^{\mathrm{w}}_{\mathrm{d}} B)$ is valid. On the other hand, one of Lewis's motivations is to avoid *paradoxes of material implication* which are valid in our system.[15] Our implication is similar to $\to_L$ with respect to the classical one, viz., it is stronger than it because $\varphi \to \psi \models_{\mathbf{S5}} \Box\varphi \to \psi$, and the opposite is false, though. This fact means that Quine's critique on $\to_L$, which roughly bases on the fact that even for **S1**, if $\varphi \to \psi$ is a theorem, then $\varphi \to_L \psi$ is a theorem, either, holds true for our implication [8].

Another suggestion by Lewis is about the classical equivalence between implication and disjunction: $A \to B =_{\mathrm{df}} \neg A \vee B$. In our logic, it fails: $\neg A \vee^{\mathrm{l}}_{\mathrm{d}} B \models A \to^{\mathrm{w}}_{\mathrm{d}} B$, and the opposite is false, though, because $\Box(\Box\neg\varphi \vee \psi) \models_{\mathbf{S5}} \Box(\Box\varphi \to \psi)$ with the opposite being false. This is in sharp contrast to Lewis, who bases upon MacColl's ideas that the failure of the above-mentioned equivalence is caused by $\neg A \vee_L B \not\models A \to_L B$, where $\vee_L$ stands for the Lewisian disjunction, while the opposite is true: "Lewis infers that disjunction too must be given a new intensional sense, according to which $(p \vee q)$ holds just in case if p were not the case it would have to be the case that q. Considerations of this sort, based on the distinction between extensional and intensional readings of the connectives, were not original to Lewis. Already [...] MacColl [...] claimed that $(p \to q)$ and $(\neg p \vee q)$ are not equivalent: $(\neg p \vee q)$ follows from $(p \to q)$, but not *vice versa*" [8].

[13]Perzanowski, who is the editor of the contemporary translation of both Jaśkowski's papers, notes: "Observe that the present criticism in comparison with the previous one, is rather weak. Some calculi of the strict implication can thereby be treated as paraconsistent ones" [23, p. 56].

[14]Recall that the same analysis shows that our implication is stronger than the Jaśkowskian one.

[15]Note that avoiding those paradoxes is not our motivation whatsoever.

4. Soundness and completeness

4.1. L-counterpart of S5

First of all, let us recall the axiomatics of the modal logic **S5**. It can be axiomatized by following axioms and rules[16]:

$$\text{All the axiom schemes of } \mathbf{CPL} \qquad \text{(CPL)}$$

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \qquad \text{(K)} \qquad\qquad \Box\varphi \to \Box\Box\varphi \qquad \text{(4)}$$

$$\Box\varphi \to \varphi \qquad \text{(T)} \qquad\qquad \Diamond\Box\varphi \to \varphi \qquad (\mathsf{B^d})$$

$$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi \qquad (\mathrm{Def}_\Diamond)$$

$$\frac{\varphi \quad \varphi \to \psi}{\psi} \qquad \text{(MP)} \qquad\qquad \frac{\varphi}{\Box\varphi} \qquad \text{(Nec)}$$

Instead of ($\mathsf{B^d}$), the formula (B) $\varphi \to \Box\Diamond\varphi$ is usually used. As it is known (see, e.g., [17, p. 44–45]), these formulas are replaceable in all normal logics.

Da Costa and Dubikajtis [14] as well as Omori and Alama [38] used the notion of "M-counterpart of **S5**" denoted as M(**S5**) (following Perzanowski's terminology [39]), where $\mathrm{M}(\mathbf{S5}) = \{\varphi \in \mathscr{F}_{\mathrm{m}} | \vdash_{\mathbf{S5}} \Diamond\varphi\}$. While changing $\Diamond$ to $\Box$ in the definitions of the discussive connectives, we also incline towards an application of the same point of view when formally explicating the point of view of an external observer in Jaśkowski's model of discussion. In the presented variant, the external observer would be more careful by accepting a given discussive thesis only when its translated modal version is necessarily accepted. That is why we use the notion of "L-counterpart of **S5**" denoted as L(**S5**) (following Perzanowski's terminology again, where $\mathrm{L}(\mathbf{S5}) = \{\varphi \in \mathscr{F}_{\mathrm{m}} | \vdash_{\mathbf{S5}} \Box\varphi\}$), in the definition of the proposed variant of discussive logic. However, observe the below Fact that follows from [39, (3.6)].

Fact 4.1. L(**S5**) = **S5**.

Let us give an axiomatization of L(**S5**) (taking into account the given above Fact 4.1, it is also an axiomatization of **S5**) corresponding to the

[16] '**CPL**' is for classical propositional logic, of course. We can consider any fixed axiomatization of **CPL** or just take all theses of **CPL**.

recalled above axiomatization of **S5** and useful for our next considerations. Thus, we present the system **JL**. It corresponds also in a way to Kotas's axiomatization of **LS5** — the set of all theses of **S5** having $\Box$ at the beginning [26]. The relation $\vdash_{\mathbf{JL}}$ is determined by the following axioms and rules of inference.

AJL_1 $\Diamond\Box\varphi$, where φ is an axiom scheme of a fixed axiomatization of **CPL**,

AJL_2 $\Diamond\Box(\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi))$

AJL_3 $\Diamond\Box(\Box\varphi \to \varphi)$

AJL_4 $\Diamond\Box(\varphi \to \Diamond\varphi)$[17]

AJL_5 $\Diamond\Box(\Box\varphi \to \Box\Box\varphi)$

AJL_6 $\Diamond\Box(\Diamond\Box\varphi \to \varphi)$

AJL_7 $\Diamond\Box(\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi)$

RJL_1 $\dfrac{\varphi \quad \Diamond\Box(\varphi \to \psi)}{\psi}$

RJL_2 $\dfrac{\varphi}{\Box\varphi}$

Lemma 4.2. *If* $\vdash_{\mathbf{JL}} \varphi$*, then* $\vdash_{\mathbf{S5}} \Box\varphi$*.*

Proof: Induction on the length of a derivation of φ in **JL**. Suppose that $\vdash_{\mathbf{JL}} \varphi$. Then φ is an axiom or is obtained by RJL_1 or RJL_2.

Let φ be an axiom. It is standard that if $\varphi \in \mathbf{CPL}$ or it is an instance of (K), (T), (T^{d}), (4), (B^{d}) or ($\mathrm{Def}_\Diamond$), then $\Box\Diamond\Box\varphi \in \mathbf{S5}$. Thus, $\vdash_{\mathbf{S5}} \Box\Diamond\Box\varphi$, i.e., $\vdash_{\mathbf{S5}} \Box\varphi$.

[17] Of course, (T^{d}): $\varphi \to \Diamond\varphi$ is derivable on basis of the given axiomatization of **S5**. Note also that the axiom AJL_4 is needed to rebuild $\Diamond\Box$ before formulae obtained by (MP) (see Lemma 4.3 below). The need for AJL_4, whose derivability on the basis of the rest of the system $\vdash_{\mathbf{JL}}$ goes beyond the scope of the paper, is connected with saving 1–1 correspondence between L(**S5**) and **JL**.

Let φ be obtained by RJL_1 from some formulas ψ and $\Diamond\Box(\psi \to \varphi)$. By the inductive hypothesis, $\vdash_{\mathbf{S5}} \Box\psi$ and $\vdash_{\mathbf{S5}} \Box\Diamond\Box(\psi \to \varphi)$. Since $\vdash_{\mathbf{S5}} \Box\Diamond\Box F \leftrightarrow \Box F$, for any formula F (see, e. g. [17, p. 43]), we have $\vdash_{\mathbf{S5}} \Box(\psi \to \varphi)$. Since $\vdash_{\mathbf{S5}} \Box(\psi \to \varphi) \to (\Box\psi \to \Box\varphi)$, we have $\vdash_{\mathbf{S5}} \Box\varphi$.

Let φ be obtained by RJL_2 from some formula ψ. Then φ has the form of $\Box\psi$. By the inductive hypothesis, $\vdash_{\mathbf{S5}} \Box\psi$. By Gödel's rule, we get $\vdash_{\mathbf{S5}} \Box\Box\psi$, i.e., $\vdash_{\mathbf{S5}} \Box\varphi$. □

Lemma 4.3. *If* $\vdash_{\mathbf{S5}} \varphi$, *then* $\vdash_{\mathbf{JL}} \Diamond\Box\varphi$.

Proof: Induction on the length of a derivation of φ in **S5**. Suppose that $\vdash_{\mathbf{S5}} \varphi$. Then φ is an axiom or is obtained by modus ponens (MP) or by Gödel's rule.

Let φ be an axiom. Then $\Diamond\Box\varphi$ is an axiom of **JL**.

Let φ be obtained by (MP) from some formulas γ and $\gamma \to \varphi$. By the inductive hypothesis, $\vdash_{\mathbf{JL}} \Diamond\Box\gamma$ and $\vdash_{\mathbf{JL}} \Diamond\Box(\gamma \to \varphi)$. Using AJL_6 (i.e., $\Diamond\Box(\Diamond\Box\gamma \to \gamma)$) and RJL_1, we get $\vdash_{\mathbf{JL}} \gamma$. Then, by RJL_1, $\vdash_{\mathbf{JL}} \varphi$. By RJL_2, $\vdash_{\mathbf{JL}} \Box\varphi$. By AJL_4, $\vdash_{\mathbf{JL}} \Diamond\Box(\Box\varphi \to \Diamond\Box\varphi)$. Hence, by RJL_1, $\vdash_{\mathbf{JL}} \Diamond\Box\varphi$.

Let φ be obtained by Gödel's rule from some formula γ. Then φ has the form of $\Box\gamma$. By the inductive hypothesis, $\vdash_{\mathbf{JL}} \Diamond\Box\gamma$. Using AJL_6 and RJL_1, we get $\vdash_{\mathbf{JL}} \gamma$. Applying RJL_2 twice, we obtain $\vdash_{\mathbf{JL}} \Box\Box\gamma$. By AJL_6 and RJL_1, $\vdash_{\mathbf{JL}} \Diamond\Box\Box\gamma$, i.e., $\vdash_{\mathbf{JL}} \Diamond\Box\varphi$. □

Lemma 4.4. $\varphi \in \mathrm{L}(\mathbf{S5})$ *iff* $\vdash_{\mathbf{JL}} \varphi$.

Proof: Suppose that $\varphi \in \mathrm{L}(\mathbf{S5})$. Then $\vdash_{\mathbf{S5}} \Box\varphi$, by the definition of $\mathrm{L}(\mathbf{S5})$. By Lemma 4.3, $\vdash_{\mathbf{JL}} \Diamond\Box\Box\varphi$. By AJL_6, $\vdash_{\mathbf{JL}} \Diamond\Box(\Diamond\Box\Box\varphi \to \Box\varphi)$. By RJL_1, $\vdash_{\mathbf{JL}} \Box\varphi$. By AJL_3, $\vdash_{\mathbf{JL}} \Diamond\Box(\Box\varphi \to \varphi)$, hence, using RJL_1, we infer $\vdash_{\mathbf{JL}} \varphi$.

Suppose that $\vdash_{\mathbf{JL}} \varphi$. Then $\vdash_{\mathbf{S5}} \Box\varphi$, by Lemma 4.2. By the definition of $\mathrm{L}(\mathbf{S5})$, $\varphi \in \mathrm{L}(\mathbf{S5})$. □

4.2. L-counterpart of S5 and paracomplete discussive logic

The system **JL** introduced above employs a modal rather than a discussive language.

Let us introduce a translation function π from $\mathscr{L}_{\mathrm{m}}$ to $\mathscr{L}_{\mathrm{r}}^*$, $\mathscr{L}_{\mathrm{l}}^*$, and $\mathscr{L}_{\mathrm{b}}^*$ (where $\perp$ denotes $p \wedge \neg p$):

- $\pi(P) = P$, for any $P \in \mathcal{P}$,
- $\pi(\neg\varphi) = \neg\pi(\varphi)$,
- $\pi(\Diamond\varphi) = \neg\pi(\varphi) \to^{\mathrm{w}}_{\mathrm{d}} \bot$,
- $\pi(\Box\varphi) = \neg(\pi(\varphi) \to^{\mathrm{w}}_{\mathrm{d}} \bot)$,
- $\pi(\varphi \wedge \psi) = \pi(\varphi) \wedge \pi(\psi)$.
- $\pi(\varphi \vee \psi) = \neg(\neg\pi(\varphi) \wedge \neg\pi(\psi))$,
- $\pi(\varphi \to \psi) = \neg(\pi(\varphi) \wedge \neg\pi(\psi))$.

We are going to use the following two axiomatizations of **S5** in the $\{\neg, \wedge, \Box\}$-language for the reasons connected with the translation π given above. We use Rosser's [44, p. 55] axiomatization of **CPL** in the $\{\neg, \wedge\}$-language.[18]

The given below consequence relation meant for **S5** is denoted by $\vdash_{\mathbf{S5}^{\neg\wedge}}$:

Ax$_1$ $\neg\big(\varphi \wedge \neg(\varphi \wedge \varphi)\big)$

Ax$_2$ $\neg\big((\varphi \wedge \psi) \wedge \neg\varphi\big)$

Ax$_3$ $\neg\big(\neg(\varphi \wedge \neg\psi) \wedge \neg\neg(\neg(\psi \wedge \gamma) \wedge \neg\neg(\gamma \wedge \varphi))\big)$

Ax$_4$ $\neg\big(\Box\neg(\varphi \wedge \neg\psi) \wedge \neg\neg(\Box\varphi \wedge \neg\Box\psi)\big)$

Ax$_5$ $\neg(\Diamond\varphi \wedge \neg\neg\Box\neg\varphi) \wedge \neg(\neg\Box\neg\varphi \wedge \neg\Diamond\varphi)$

Ax$_6$ $\neg(\Box\varphi \wedge \neg\varphi)$

Ax$_7$ $\neg(\Box\varphi \wedge \neg\Box\Box\varphi)$

Ax$_8$ $\neg(\Diamond\Box\varphi \wedge \neg\varphi)$

RS5$_1$ $\dfrac{\varphi \quad \neg(\varphi \wedge \neg\psi)}{\psi}$ RS5$_2$ $\dfrac{\varphi}{\Box\varphi}$

Let us denote the function from $\mathscr{L}_{\mathrm{m}}$ to $\mathscr{L}_{\mathrm{m}}$ by δ that operates as π for $\neg$, $\wedge$, $\vee$, and $\to$, while for modal operators we assume that $\delta(\Diamond\varphi) = \Diamond\delta(\varphi)$ and $\delta(\Box\varphi) = \Box\delta(\varphi)$. We have an easy-to-see:

[18] The original Rosserian axioms look as follows: (1) $P \supset PP$, (2) $PQ \supset P$, (3) $P \supset Q. \supset .{\sim}(QR) \supset {\sim}(RP)$. Note that due to the invalidity of $(A \to^{\mathrm{w}}_{\mathrm{d}} B) \to^{\mathrm{w}}_{\mathrm{d}} \neg(A \wedge \neg B)$ on the basis of $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$, one cannot interpret implication in the Rosserian axiomatization as $\to^{\mathrm{w}}_{\mathrm{d}}$.

FACT 4.5. For any formula $\varphi \in \mathscr{F}_{\mathrm{m}}$

1. $\varphi \in \mathbf{S5}$ iff $\vdash_{\mathbf{S5}^{\neg\wedge}} \delta(\varphi)$,
2. if φ is expressed in the language with $\{\Diamond, \Box, \neg, \wedge\}$ and $\vdash_{\mathbf{S5}^{\neg\wedge}} \varphi$, then $\varphi \in \mathbf{S5}$.

We will use another consequence relation denoted as $\vdash_{\mathbf{JL}^{\neg\wedge}}$ that corresponds to $\vdash_{\mathbf{JL}}$.

AJL$_1$ $\Diamond\Box\neg\big(\varphi \wedge \neg(\varphi \wedge \varphi)\big)$

AJL$_2$ $\Diamond\Box\neg\big((\varphi \wedge \psi) \wedge \neg\varphi\big)$

AJL$_3$ $\Diamond\Box\neg\big(\neg(\varphi \wedge \neg\psi) \wedge \neg\neg(\neg(\psi \wedge \gamma) \wedge \neg\neg(\gamma \wedge \varphi))\big)$

AJL$_4$ $\Diamond\Box\neg\big(\Box\neg(\varphi \wedge \neg\psi) \wedge \neg\neg(\Box\varphi \wedge \neg\Box\psi)\big)$

AJL$_5$ $\Diamond\Box(\neg(\Diamond\varphi \wedge \neg\neg\Box\neg\varphi) \wedge \neg(\neg\Box\neg\varphi \wedge \neg\Diamond\varphi))$

AJL$_6$ $\Diamond\Box\neg(\Box\varphi \wedge \neg\varphi)$

AJL$_7$ $\Diamond\Box\neg(\varphi \wedge \neg\Diamond\varphi)$

AJL$_8$ $\Diamond\Box\neg(\Box\varphi \wedge \neg\Box\Box\varphi)$

AJL$_9$ $\Diamond\Box\neg(\Diamond\Box\varphi \wedge \neg\varphi)$

RJL$_1$ $\dfrac{\varphi \quad \Diamond\Box\neg(\varphi \wedge \neg\psi)}{\psi}$ RJL$_2$ $\dfrac{\varphi}{\Box\varphi}$

FACT 4.6. For any formula φ in the language with $\Diamond, \Box, \neg, \wedge$:

$$\vdash_{\mathbf{JL}} \varphi \text{ iff } \vdash_{\mathbf{JL}^{\neg\wedge}} \varphi$$

LEMMA 4.7. *The following rule is inferable on the basis of* $\mathbf{D_2^{\mathsf{p}}}$*:*

$$\frac{D}{\neg\neg(D \to^{\mathrm{w}}_{\mathrm{d}} \bot) \to^{\mathrm{w}}_{\mathrm{d}} \bot} \qquad (\Diamond\Box^{\pi})$$

PROOF:

1. D assumption
2. $\neg\neg(D \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)$ premiss
3. $\neg\neg(D \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} (D \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot)$ Ax_6
4. $D \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$ $(\mathrm{MP}^{\mathrm{w}}_{\mathrm{d}})$: 2, 3
5. $\bot$ $(\mathrm{MP}^{\mathrm{w}}_{\mathrm{d}})$: 1, 4
6. $D \vdash_{\mathbf{D}_2^{\mathrm{p}}} \neg\neg(D \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$ Lemma 3.1: 1–5

□

LEMMA 4.8. *For any axiom Ax of $\vdash_{\mathbf{JL}^{\neg\wedge}}$, $\vdash_{\mathbf{D}_2^{\mathrm{p}}} \pi(Ax)$.*

PROOF:

- The case of AJL_1: $\Diamond\Box\neg\big(\varphi \wedge \neg(\varphi \wedge \varphi)\big)$.

  Since $\pi(\Diamond\Box\neg\big(\varphi\wedge\neg(\varphi\wedge\varphi)\big)) = \neg\neg\Big(\neg\big(\pi(\varphi)\wedge\neg(\pi(\varphi)\wedge\pi(\varphi))\big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot\Big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$, we apply $(\Diamond\Box^{\pi})$ for $D = \neg\big(\pi(\varphi) \wedge \neg(\pi(\varphi) \wedge \pi(\varphi))\big)$—an instance of Ax_3.

- The case of AJL_2: $\Diamond\Box\neg\big((\varphi \wedge \psi) \wedge \neg\varphi\big)$.

  Since $\pi\big(\Diamond\Box\neg((\varphi\wedge\psi)\wedge\neg\varphi)\big) = \neg\neg\Big(\neg\big((\pi(\varphi)\wedge\pi(\psi))\wedge\neg\pi(\varphi)\big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot\Big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$, we apply $(\Diamond\Box^{\pi})$ for $D = \neg\big((\pi(\varphi) \wedge \pi(\psi)) \wedge \neg\pi(\varphi)\big)$—an instance of Ax_4.

- The case of AJL_3: $\Diamond\Box\neg\big(\neg(\varphi \wedge \neg\psi) \wedge \neg\neg(\neg(\psi \wedge \gamma) \wedge \neg\neg(\gamma \wedge \varphi))\big)$.

  Since $\pi\big(\Diamond\Box\neg(\neg(\varphi\wedge\neg\psi)\wedge\neg\neg(\neg(\psi\wedge\gamma)\wedge\neg\neg(\gamma\wedge\varphi)))\big) = \neg\neg\Big(\neg\big(\neg(\pi(\varphi) \wedge \neg\pi(\psi)) \wedge \neg\neg(\neg(\pi(\psi) \wedge \pi(\gamma)) \wedge \neg\neg(\pi(\gamma) \wedge \pi(\varphi)))\big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot\Big) \rightarrow^{\mathrm{w}}_{\mathrm{d}} \bot$, we apply $(\Diamond\Box^{\pi})$ for $D = \neg\big(\neg(\pi(\varphi) \wedge \neg\pi(\psi)) \wedge \neg\neg(\neg(\pi(\psi) \wedge \pi(\gamma)) \wedge \neg\neg(\pi(\gamma) \wedge \pi(\varphi)))\big)$—an instance of Ax_5.

- The case of AJL$_4$: $\Diamond\Box\neg\big(\neg\Box(\varphi\wedge\neg\psi)\wedge\neg\neg(\Box\varphi\wedge\neg\Box\psi)\big)$.

  Since $\pi\big(\Diamond\Box\neg(\neg\Box(\varphi\wedge\neg\psi)\wedge\neg\neg(\Box\varphi\wedge\neg\Box\psi))\big) = \neg\neg\Big(\neg\big(\neg\neg((\pi(\varphi)\wedge\neg\pi(\psi))\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\wedge\neg\neg(\neg(\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\wedge\neg\neg(\pi(\psi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot))\big)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot\Big)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot$, we apply ($\Diamond\Box^{\pi}$) for $D = \neg\big(\neg\neg((\pi(\varphi)\wedge\neg\pi(\psi))\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\wedge\neg\neg(\neg(\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\wedge\neg\neg(\pi(\psi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot))\big)$—an instance of Ax$_9$.

- The case of AJL$_5$: $\Diamond\Box(\neg(\Diamond\varphi\wedge\neg\neg\Box\neg\varphi)\wedge\neg(\neg\Box\neg\varphi\wedge\neg\Diamond\varphi))$.

  Since $\pi(\Diamond\Box(\neg(\Diamond\varphi\wedge\neg\neg\Box\neg\varphi)\wedge\neg(\neg\Box\neg\varphi\wedge\neg\Diamond\varphi))) = \neg\neg\Big((\neg((\neg\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\wedge\neg\neg\neg(\neg\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot))\wedge\neg(\neg\neg(\neg\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\wedge\neg(\neg\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)))\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot\Big)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot$. We apply ($\Diamond\Box^{\pi}$) for $D = \big(\neg((\neg\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\wedge\neg\neg\neg(\neg\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot))\wedge\neg(\neg\neg(\neg\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\wedge\neg(\neg\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot))\big)$—an instance of classical thesis that is inferable by Fact 3.2.

- The case of AJL$_6$: $\Diamond\Box\neg(\Box\varphi\wedge\neg\varphi)$.

  Since $\pi(\Diamond\Box\neg(\Box\varphi\wedge\neg\varphi)) = \neg\neg\big(\neg(\neg(\pi(\varphi)\rightarrow\bot)\wedge\neg\pi(\varphi))\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot\big)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot$, we apply ($\Diamond\Box^{\pi}$) for $D = \neg(\neg(\pi(\varphi)\rightarrow\bot)\wedge\neg\pi(\varphi))$—an instance of Ax$_7$.

- The case of AJL$_7$: $\Diamond\Box\neg(\varphi\wedge\neg\Diamond\varphi)$

  Since $\pi(\Diamond\Box\neg(\varphi\wedge\neg\Diamond\varphi)) = \neg\neg\big(\neg(\pi(\varphi)\wedge\neg(\neg\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot))\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot\big)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot$, we apply ($\Diamond\Box^{\pi}$) for $D = \neg(\pi(\varphi)\wedge\neg(\neg\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot))$—following from Ax$_7$ by Fact 3.2.

- The case of AJL$_8$: $\Diamond\Box\neg(\Box\varphi\wedge\neg\Box\Box\varphi)$.

  Since $\pi(\Diamond\Box\neg(\Box\varphi\wedge\neg\Box\Box\varphi)) = \neg\neg\Big(\neg\big(\neg(\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\wedge\neg\neg(\neg(\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\big)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot\Big)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot$, we apply ($\Diamond\Box^{\pi}$) for $D = \neg\big(\neg(\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\wedge\neg\neg(\neg(\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\big)$—an instance of Ax$_{10}$.

- The case of AJL$_9$: $\Diamond\Box\neg(\Diamond\Box\varphi\wedge\neg\varphi)$.

  Since $\Diamond\Box\neg(\Diamond\Box\varphi\wedge\neg\varphi) = \neg\neg\Big(\neg\big((\neg\neg(\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\wedge\neg\pi(\varphi)\big)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot\Big)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot$, we apply ($\Diamond\Box^{\pi}$) for $D = \neg\big((\neg\neg(\pi(\varphi)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\rightarrow^{\mathrm{w}}_{\mathrm{d}}\bot)\wedge\neg\pi(\varphi)\big)$—an instance of Ax$_8$. $\Box$

Now we want to show that in $\mathbf{D}_2^{\mathrm{p}}$, the specific discussive connectives $\to_{\mathrm{d}}^{\mathrm{w}}$ and $\vee_{\mathrm{d}}^{\mathrm{l}}$ are properly characterized. Such characterizations are in the form of discussive implications in both directions.

Lemma 4.9. *For any $C \in \mathscr{L}_{\mathrm{l}}^{*}$, it holds $\vdash_{\mathbf{D}_2^{\mathrm{p}}} \pi(\sigma(C)) \to_{\mathrm{d}}^{\mathrm{w}} C$.*

Proof: First, we show that $\vdash_{\mathbf{D}_2^{\mathrm{p}}} \neg(\pi(\sigma(A)) \wedge \neg A)$. To obtain that for any A, $\vdash_{\mathbf{D}_2^{\mathrm{p}}} \neg(\pi(\sigma(A)) \wedge \neg A)$, we can prove $\neg(\pi(\sigma(A)) \wedge \neg A)$ and additionally $\neg(A \wedge \neg\pi(\sigma(A)))$ using simultaneous induction on the construction of A.

The case when A is a variable is trivial due to the fact that (Ax$_3$)–(Ax$_5$) with (MP) expressed for the language $\{\wedge, \neg\}$ constitute the complete axiomatization of classical logic. Similarly, due to the fact that $\pi(\sigma(\neg B)) = \neg\pi(\sigma(B))$ and $\pi(\sigma(B \wedge C)) = \pi(\sigma(B)) \wedge \pi(\sigma(C))$, the cases of $\wedge$ and $\neg$ follow by inductive hypotheses for B and C, and extensionality for classical logic expressed in $\{\wedge, \neg\}$.

Case $A = B \to_{\mathrm{d}}^{\mathrm{w}} C$.

By definitions $\neg\big(\pi(\sigma(B \to_{\mathrm{d}}^{\mathrm{w}} C)) \wedge \neg(B \to_{\mathrm{d}}^{\mathrm{w}} C)\big) = \neg\big(\pi(\Box\sigma(B) \to \sigma(C)) \wedge \neg(B \to_{\mathrm{d}}^{\mathrm{w}} C)\big) = \neg\big(\neg(\neg(\pi(\sigma(B)) \to_{\mathrm{d}}^{\mathrm{w}} \bot) \wedge \neg\pi(\sigma(C))) \wedge \neg(B \to_{\mathrm{d}}^{\mathrm{w}} C)\big)$ and $\neg\big((B \to_{\mathrm{d}}^{\mathrm{w}} C) \wedge \neg\pi(\sigma(B \to_{\mathrm{d}}^{\mathrm{w}} C))\big) = \neg\big((B \to_{\mathrm{d}}^{\mathrm{w}} C) \wedge \neg\pi(\Box\sigma(B) \to \sigma(C))\big) = \neg\big((B \to_{\mathrm{d}}^{\mathrm{w}} C) \wedge \neg\neg(\neg(\pi(\sigma(B)) \to_{\mathrm{d}}^{\mathrm{w}} \bot) \wedge \neg\pi(\sigma(C)))\big)$.

Consider the following inference.

1. $\neg(B \wedge \neg\pi(\sigma(B)))$ — inductive hypothesis
2. $\neg(B \wedge \neg\pi(\sigma(B))) \to_{\mathrm{d}}^{\mathrm{w}} \neg(\neg(B \wedge \neg\pi(\sigma(B))) \to_{\mathrm{d}}^{\mathrm{w}} \bot)$ — Ax$_{16}$
3. $\neg(\neg(B \wedge \neg\pi(\sigma(B))) \to_{\mathrm{d}}^{\mathrm{w}} \bot)$ — 1, 2 and (MP$_{\mathrm{d}}^{\mathrm{w}}$)
4. $\neg\big(\neg(\neg(B \wedge \neg\pi(\sigma(B))) \to_{\mathrm{d}}^{\mathrm{w}} \bot) \wedge \neg\neg(\neg(B \to_{\mathrm{d}}^{\mathrm{w}} \bot) \wedge \neg\neg(\pi(\sigma(B)) \to_{\mathrm{d}}^{\mathrm{w}} \bot))\big)$ — Ax$_9$
5. $\neg(\neg(B \wedge \neg\pi(\sigma(B))) \to_{\mathrm{d}}^{\mathrm{w}} \bot) \to_{\mathrm{d}}^{\mathrm{w}} \neg(\neg(B \to_{\mathrm{d}}^{\mathrm{w}} \bot) \wedge \neg\neg(\pi(\sigma(B)) \to_{\mathrm{d}}^{\mathrm{w}} \bot))$ — 4, Ax$_{17}$ and (MP$_{\mathrm{d}}^{\mathrm{w}}$)
6. $\neg(\neg(B \to_{\mathrm{d}}^{\mathrm{w}} \bot) \wedge \neg\neg(\pi(\sigma(B)) \to_{\mathrm{d}}^{\mathrm{w}} \bot))$ — 3, 5, and (MP$_{\mathrm{d}}^{\mathrm{w}}$)

Next, applying the above inferred formula, an instance of the axiom Ax$_{11}$: $\neg\big(\neg\big(\neg(B \to_{\mathrm{d}}^{\mathrm{w}} \bot) \wedge \neg C\big) \wedge \neg(B \to_{\mathrm{d}}^{\mathrm{w}} C)\big)$, the inductive hypothesis for C: $\neg(\pi(\sigma(C)) \wedge \neg C)$, and classical logic expressed in $\{\wedge, \neg\}$ (due to Fact 3.2) we obtain the required thesis $\neg\big(\pi(\sigma(B \to_{\mathrm{d}}^{\mathrm{w}} C)) \wedge \neg(B \to_{\mathrm{d}}^{\mathrm{w}} C)\big)$.

For the case of $\neg\big((B \to^{\mathrm{w}}_{\mathrm{d}} C) \wedge \neg\pi(\sigma(B \to^{\mathrm{w}}_{\mathrm{d}} C))\big)$, consider the following inference.

1. $\neg(\pi(\sigma(B)) \wedge \neg B)$ inductive hypothesis
2. $\neg(\pi(\sigma(B)) \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \neg(\neg(\pi(\sigma(B)) \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \bot)$ Ax$_{16}$
3. $\neg(\neg(\pi(\sigma(B)) \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \bot)$ 1, 2 and (MP$^{\mathrm{w}}_{\mathrm{d}}$)
4. $\neg\big(\neg(\neg(\pi(\sigma(B)) \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\neg(\pi(\sigma(B)) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))\big)$ Ax$_9$
5. $\neg(\neg(\pi(\sigma(B)) \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \to^{\mathrm{w}}_{\mathrm{d}} \neg(\neg(\pi(\sigma(B)) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))$ 4, Ax$_{17}$ and (MP$^{\mathrm{w}}_{\mathrm{d}}$)
6. $\neg(\neg(\pi(\sigma(B)) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))$ 3, 5, and (MP$^{\mathrm{w}}_{\mathrm{d}}$)

Similarly, using the obtained formula, Ax$_{12}$, the inductive hypothesis for C: $\neg(C \wedge \neg\pi(\sigma(C)))$ and classical logic expressed in $\{\wedge, \neg\}$ we obtain the required formula.

Case $A = B \vee^{\mathrm{l}}_{\mathrm{d}} C$.

By definitions $\neg\big(\pi(\sigma(B \vee^{\mathrm{l}}_{\mathrm{d}} C)) \wedge \neg(B \vee^{\mathrm{l}}_{\mathrm{d}} C)\big) = \neg\big(\pi(\Box\sigma(B) \vee \sigma(C)) \wedge \neg(B \vee^{\mathrm{l}}_{\mathrm{d}} C)\big) = \neg\big(\neg(\neg\neg(\pi(\sigma(B)) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\pi(\sigma(C))) \wedge \neg(B \vee^{\mathrm{l}}_{\mathrm{d}} C)\big)$.

Again, taking into account $\neg(\neg(\pi(\sigma(B)) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))$ and $\neg(\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\pi(\sigma(B)) \to^{\mathrm{w}}_{\mathrm{d}} \bot))$, applying an instance of the axiom Ax$_{14}$ $\neg\big(\neg(\neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg C) \wedge \neg(B \vee^{\mathrm{l}}_{\mathrm{d}} C)\big)$, the inductive hypothesis for C, and extensionality for classical logic expressed in $\{\wedge, \neg\}$ (due to Fact 3.2) we obtain the required thesis.

The case of $\neg\big((B \vee^{\mathrm{l}}_{\mathrm{d}} C) \wedge \neg\pi(\sigma(B \vee^{\mathrm{l}}_{\mathrm{d}} C))\big)$ is being treated analogously with the help of Ax$_{13}$.

Having proved $\vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \neg(\pi(\sigma(A)) \wedge \neg A)$, the required thesis follows by Ax$_{17}$. □

We need an additional, easy-to-see fact.

Fact 4.10. For any $\varphi \in \mathscr{F}_{\mathrm{m}}$, it holds $\pi(\varphi) = \pi(\delta(\varphi))$.

Theorem 4.11. *For any formula A of the discussive language:*

$$A \in \mathbf{D}^{\mathsf{p}}_{\mathbf{2}} \text{ iff } \vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} A$$

PROOF: Assume that $A \in \mathbf{D_2^p}$. By definitions, $\Box\sigma(A) \in \mathbf{S5}$, while by Fact 4.5 and definition of δ, $\vdash_{\mathbf{S5}^{\neg\wedge}} \Box\delta(\sigma(A))$, and again by Fact 4.5, $\Box\delta(\sigma(A)) \in \mathbf{S5}$. Hence, by Lemma 4.4 we have $\vdash_{\mathbf{JL}} \delta(\sigma(A))$. By Fact 4.6, $\vdash_{\mathbf{JL}^{\neg\wedge}} \delta(\sigma(A))$. Consider a respective proof $\varphi_1, \ldots, \varphi_n = \delta(\sigma(A))$, on the basis of $\vdash_{\mathbf{JL}^{\neg\wedge}}$. Now let us consider $C_1 = \pi(\varphi_1), \ldots, C_n = \pi(\varphi_n) = \pi(\delta(\sigma(A)))$, $C_{n+1} = \pi(\delta(\sigma(A))) \to_{\mathrm{d}}^{\mathrm{w}} A$, $C_{n+2} = A$. Using Fact 4.10, we see that $C_n = \pi(\sigma(A))$, $C_{n+1} = \pi(\delta(\sigma(A))) \to_{\mathrm{d}}^{\mathrm{w}} A = \pi(\sigma(A)) \to_{\mathrm{d}}^{\mathrm{w}} A$. By induction on the length of the proof, we show that for each $1 \leqslant i \leqslant n+2$, $\vdash_{\mathbf{D_2^p}} C_i$. The case of axioms follows by Lemma 4.8.

Consider the cases of rules. Assume that φ_i, where $1 \leqslant i \leqslant n$ results from an application of RJL_1, that is, there are $1 < j, k < i$ such that $\varphi_k = \Diamond\Box\neg(\varphi_j \wedge \neg\varphi_i)$. By inductive hypothesis $\vdash_{\mathbf{D_2^p}} \pi(\varphi_j)$ and $\vdash_{\mathbf{D_2^p}} \pi(\Diamond\Box\neg(\varphi_j \wedge \neg\varphi_i))$, i.e., $\vdash_{\mathbf{D_2^p}} \neg\neg(\neg(\pi(\varphi_j) \wedge \neg\pi(\varphi_i)) \to_{\mathrm{d}}^{\mathrm{w}} \bot) \to_{\mathrm{d}}^{\mathrm{w}} \bot$. Consider the following sequence:

1. $\pi(\varphi_j)$ — by inductive hypothesis
2. $\neg\neg(\neg(\pi(\varphi_j) \wedge \neg\pi(\varphi_i)) \to_{\mathrm{d}}^{\mathrm{w}} \bot) \to_{\mathrm{d}}^{\mathrm{w}} \bot$ — by inductive hypothesis
3. $\pi(\varphi_j) \to_{\mathrm{d}}^{\mathrm{w}} \Big((\neg\neg(\neg(\pi(\varphi_j) \wedge \neg\pi(\varphi_i)) \to_{\mathrm{d}}^{\mathrm{w}} \bot) \to_{\mathrm{d}}^{\mathrm{w}} \bot) \to_{\mathrm{d}}^{\mathrm{w}} \pi(\varphi_i)\Big)$ — Ax_{15}
4. $\pi(\varphi_i)$ — $2 \times (\mathrm{MP}_{\mathrm{d}}^{\mathrm{w}})$: 1, 2, 3

Assume that φ_i, where $1 \leqslant i \leqslant n$ results from an application of RJL_2, that is, there is $1 < k < i$ such that $\varphi_i = \Box\varphi_j$. We have to show that $\vdash_{\mathbf{D_2^p}} \pi(\Box\varphi_j)$, i.e., $\vdash_{\mathbf{D_2^p}} \neg(\pi(\varphi_j) \to_{\mathrm{d}}^{\mathrm{w}} \bot)$. By the inductive hypothesis $\vdash_{\mathbf{D_2^p}} \pi(\varphi_j)$. Consider the following sequence:

1. $\pi(\varphi_j)$ — by inductive hypothesis
2. $\pi(\varphi_j) \to_{\mathrm{d}}^{\mathrm{w}} \neg(\pi(\varphi_j) \to_{\mathrm{d}}^{\mathrm{w}} \bot)$ — Ax_{16}
3. $\neg(\pi(\varphi_j) \to_{\mathrm{d}}^{\mathrm{w}} \bot)$ — $(\mathrm{MP}_{\mathrm{d}}^{\mathrm{w}})$: 1, 2

For the last two elements in $C_1, \ldots, C_n, C_{n+1}, C_{n+2} = A$ we use Lemma 4.9 and $(\mathrm{MP}_{\mathrm{d}}^{\mathrm{w}})$.

The reverse implication is obtained by routine checking. □

5. A modification of axiomatization

Now let us consider only the $\{\neg, \to^{\mathrm{w}}_{\mathrm{d}}, \wedge\}$-part of $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$.[19] As the only rule of inference, let $\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}}$ denote the consequence relation as determined by (MP$^{\mathrm{w}}_{\mathrm{d}}$) together with the axiom schemes listed below: Ax$_1$–Ax$_5$, Ax$_{11}$, Ax$_{16}$–Ax$_{17}$, and

$$\neg((A \to^{\mathrm{w}}_{\mathrm{d}} B) \wedge (\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg B)) \tag{Imp$^{\mathsf{cl}}$}$$

$$\neg(((A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg A) \tag{B$^{\mathsf{d}}$}$$

One can easily see that by Ax$_{16}$, Ax$_{17}$, positive logic expressed with $\to^{\mathrm{w}}_{\mathrm{d}}$, and classical logic in $\neg$ and $\wedge$, and (Imp$^{\mathsf{cl}}$) we have:

Fact 5.1.

$$\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \neg(A \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} (A \to^{\mathrm{w}}_{\mathrm{d}} \neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot)) \tag{K}$$

$$\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} (A \to^{\mathrm{w}}_{\mathrm{d}} B) \to^{\mathrm{w}}_{\mathrm{d}} \neg(\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg B) \tag{Imp}$$

Lemma 5.2.

$$\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \mathrm{Ax}_9$$

$$\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \mathrm{Ax}_{12}$$

Proof: By (K), (Imp), positive logic for $\to^{\mathrm{w}}_{\mathrm{d}}$ we have:

- $\neg(A \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} (A \to^{\mathrm{w}}_{\mathrm{d}} \neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot)$

- $(A \to^{\mathrm{w}}_{\mathrm{d}} \neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot) \to^{\mathrm{w}}_{\mathrm{d}} \neg(\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))$

- $\neg(A \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \neg(\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))$

- $\neg\big(\neg(\neg(A \wedge \neg B) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(B \to^{\mathrm{w}}_{\mathrm{d}} \bot))\big)$,

The case of Ax$_{12}$ is obvious from (Imp$^{\mathsf{cl}}$) and classical logic expressed in the language with $\neg$ and $\wedge$. □

[19] Since $\vee^{\mathrm{l}}_{\mathrm{d}}$ is definable in the considered language, for the language with $\vee^{\mathrm{l}}_{\mathrm{d}}$, one could just add two axioms: $\neg((A \vee^{\mathrm{l}}_{\mathrm{d}} B) \wedge ((A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg B))$, $\neg(\neg(A \vee^{\mathrm{l}}_{\mathrm{d}} B) \wedge \neg((A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg B))$.

LEMMA 5.3. *For any formula C in the language with $\{\neg, \to^{\mathrm{w}}_{\mathrm{d}}, \wedge\}$, it holds* $\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \pi(\sigma(C)) \to^{\mathrm{w}}_{\mathrm{d}} C$.

PROOF: First, we show that $\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \neg(\pi(\sigma(A)) \wedge \neg A)$. To obtain that for any A, $\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \neg(\pi(\sigma(A)) \wedge \neg A)$, we can prove $\neg(\pi(\sigma(A)) \wedge \neg A)$ and additionally $\neg(A \wedge \neg\pi(\sigma(A)))$ using simultaneous induction on the construction of A.

The cases of a variable, $\neg$ and $\wedge$ are being handled in the same way as in the proof of Lemma 4.9.

Using the axioms Ax_{11}, Ax_{16}, and Ax_{17}, as well as Ax_{9} and Ax_{12} (inferable by Lemma 5.2), we can repeat the proof of Lemma 4.9 in its part for the case of $\to^{\mathrm{w}}_{\mathrm{d}}$.

Having proved $\vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} \neg(\pi(\sigma(A)) \wedge \neg A)$, the required thesis follows by Ax_{17}. □

THEOREM 5.4. *For any formula A of the discussive language:*

$$A \in \mathbf{D}^{\mathsf{p}}_{\mathbf{2}} \textit{ iff } \Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} A$$

PROOF: Assume that $A \in \mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$. By definition, $\Box\sigma(A) \in \mathbf{S5}$, so also $\sigma(A) \in \mathbf{S5}$. By Fact 4.5 $\vdash_{\mathbf{S5}^{\neg\wedge}} \delta(\sigma(A))$. There is a proof $\varphi_1, \ldots, \varphi_n = \delta(\sigma(A))$ on the basis of the relation $\vdash_{\mathbf{S5}^{\neg\wedge}}$. Now we consider a sequence $C_1 = \pi(\varphi_1)$, $\ldots$, $C_n = \pi(\varphi_n) = \pi(\delta(\sigma(A)))$, $C_{n+1} = \pi(\delta(\sigma(A))) \to^{\mathrm{w}}_{\mathrm{d}} A$, $C_{n+2} = A$. By Fact 4.10, we see that $C_n = \pi(\sigma(A))$ and $C_{n+1} = \pi(\sigma(A)) \to^{\mathrm{w}}_{\mathrm{d}} A$.

By induction on the length of the proof, we show that for each $1 \leqslant i \leqslant n+2$, $\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}} C_i$. For the case of an axiom scheme $\mathrm{Ax} \in \{\mathrm{Ax}_1, \mathrm{Ax}_2, \mathrm{Ax}_3\}$, we have $\pi(\mathrm{Ax})$ is an instance of an axiom scheme of $\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}}$. For the case of Ax_4, we see that $\pi\big(\neg\big(\Box\neg(\varphi\wedge\neg\psi)\wedge\neg\neg(\Box\varphi\wedge\neg\Box\psi)\big)\big) = \neg\big(\neg(\neg(\pi(\varphi)\wedge\neg\pi(\psi)) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\neg(\pi(\varphi) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\pi(\psi) \to^{\mathrm{w}}_{\mathrm{d}} \bot))\big)$. Thus, by Lemma 5.2, the required thesis is an instance of a formula provable on the basis of $\Vdash_{\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}}$.

For the case of Ax_6, we see that $\pi(\neg(\Box\varphi \wedge \neg\varphi)) = \neg(\neg(\pi(\varphi) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\pi(\varphi))$ But this follows from the thesis $A \to^{\mathrm{w}}_{\mathrm{d}} A$ and (Imp).

For the case of Ax_7, we have $\pi(\neg(\Box\varphi \wedge \neg\Box\Box\varphi)) = \neg\big(\neg(\pi(\varphi) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg(\neg(\pi(\varphi) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \to^{\mathrm{w}}_{\mathrm{d}} \bot)\big)$. By Ax_{16} we have $A \to^{\mathrm{w}}_{\mathrm{d}} \neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot)$ and $\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \to^{\mathrm{w}}_{\mathrm{d}} \neg(\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \to^{\mathrm{w}}_{\mathrm{d}} \bot)$, hence $A \to^{\mathrm{w}}_{\mathrm{d}} \neg(\neg(A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \to^{\mathrm{w}}_{\mathrm{d}} \bot)$, so the required formula follows by (Imp).

For the case of Ax_8, we have $\pi(\neg(\varphi \wedge \neg\Box\Diamond\varphi)) = \neg\big(\pi(\varphi) \wedge \neg\neg((\neg\pi(\varphi) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \to^{\mathrm{w}}_{\mathrm{d}} \bot)\big)$. However, by $(\mathrm{B}^{\mathrm{d}})$ we have $\neg(((\neg A \to^{\mathrm{w}}_{\mathrm{d}} \bot) \to^{\mathrm{w}}_{\mathrm{d}} \bot) \wedge \neg\neg A)$, hence by classical logic expressed in $\{\neg, \wedge\}$, Ax_{17}, by $\mathrm{MP}^{\mathrm{w}}_{\mathrm{d}}$ we obtain $\neg\big(A \wedge$

$\neg\neg((\neg A \to_{\mathrm{d}}^{\mathrm{w}} \bot) \to_{\mathrm{d}}^{\mathrm{w}} \bot))$, so the required scheme is an instance of the last scheme.

Consider the rule cases. Assume that φ_i, where $1 \leqslant i \leqslant n$ results from application of $\mathrm{RS5}_1$, that is, there are $1 < j, k < i$ such that $\varphi_k = \neg(\varphi_j \wedge \neg\varphi_i)$. By inductive hypothesis $\Vdash_{\mathbf{D}_2^{\mathrm{p}}} \pi(\varphi_j)$ and $\Vdash_{\mathbf{D}_2^{\mathrm{p}}} \pi(\neg(\varphi_j \wedge \neg\varphi_i))$, i.e., $\Vdash_{\mathbf{D}_2^{\mathrm{p}}} (\neg(\pi(\varphi_j) \wedge \neg\pi(\varphi_i)))$, but by Ax_{17} $\Vdash_{\mathbf{D}_2^{\mathrm{p}}} \pi(\varphi_j) \to_{\mathrm{d}}^{\mathrm{w}} \pi(\varphi_i)$, so the required formula follows by $(\mathrm{MP}_{\mathrm{d}}^{\mathrm{w}})$. The case of $\mathrm{RS5}_2$ is a direct consequence of the application of Ax_{16} and $(\mathrm{MP}_{\mathrm{d}}^{\mathrm{w}})$.

For the formula C_{n+1} we use Lemma 5.3, while C_{n+2} is obtained by the application of $(\mathrm{MP}_{\mathrm{d}}^{\mathrm{w}})$.

The fact that if $\Vdash_{\mathbf{D}_2^{\mathrm{p}}} A$, then $A \in \mathbf{D}_2^{\mathrm{p}}$ follows by routine checking. □

6. Related work

Arguably, Akama, Abe, and Nakamatsu's discursive logic is the first paracomplete discussive logic [2] that Jaśkowski's discussive logic inspires.[20] Being based on Nelson's constructive logic with a strong negation **N4** [4, 36] (the name **N4** is due to Wansing [52]), Akama et al. propose **CDLSN**, *constructive discursive logic with strong negation*, where "discursive negation is defined similar to intuitionistic negation and discursive implication is defined as material implication using discursive negation [2, p. 395] [...] **CDLSN** can be defined in two ways. One is to extend **N4** with discursive negation $\neg_{\mathrm{d}}$. The other is to weaken intuitionistic negation in **N4**. We adopt the first approach [...] Intuitionistic negation is not a discursive negation" [2, p. 398]. Below, we highlight some (dis)similarities between this and our approaches.

First, Akama et al.'s approach is not standard because it unemploys a classically-based modal logic: "Most works on discursive logic utilize classical logic and **S5** as a basis. However, we do not think that these are essential. For instance, an intuitionist hopes to have a discursive system in a constructive setting" [2, p. 397]. However, they argue that **CDLSN** is a discussive logic (see [2, pp. 406–407]). Our motivation is not to set up a discussive logic by any means. Rather, we would like to show that *non-discussive* logics are obtainable if one sticks to the standard approach for setting up discussive logic on the basis of a classically-based modal

[20] The below-mentioned exposition of both **CDLSN** itself and the ideas beyond it does not claim completeness. The reader is consulted to address [2] for details.

logic and at the same time employs a certain non-standard interpretation of discussive connectives. In particular, the non-discussive logics in this paper are obtained *via* non-standard interpretations of discussive disjunction. Alternatively and quite analogously, the non-standard interpretation of discussive negation as $\neg_{\mathrm{d}} A =_{\mathrm{df}} \Box\neg A$, which we briefly outline in Section 3 above might be employed to set up non-discussive logic. We do not get into details here, for it deserves a separate paper.

Second, both logics are similar because no version of *the law of excluded middle* is valid. Hence, both of them are paracomplete.[21] However, the invalidity of these versions stems from different reasons. The intuitionistic-like version, $A \vee {\sim} A$ ($\sim$ is the original notation in [2] for the strong negation), as well as the discussive one, $A \vee \neg_{\mathrm{d}} A$, have intuitionistic disjunction and are **CDLSN**-invalid due to the well-known properties of the given intuitionistic(-like) negations and disjunction. Our version, $A \vee_{\mathrm{d}} \neg A$, is quite opposite in a sense that it contains classical negation and discussive disjunction. And its invalidity is due only to the interpretation of discussive disjunction as $A \vee_{\mathrm{d}} \neg A =_{\mathrm{df}} \Box A \vee \neg A$, where $\Box A \vee \neg A$ is **S5**-invalid.[22]

Third, with regard to discussive implications in both logics, one of ours proves each formula from the classical implicative fragment, which is obviously not in line with Akama et al.'s intuitionistic-like motivation. Hence, $((A \rightarrow^{\mathrm{w}}_{\mathrm{d}} B) \rightarrow^{\mathrm{w}}_{\mathrm{d}} A) \rightarrow^{\mathrm{w}}_{\mathrm{d}} A$ is a theorem in our logic only.[23] Moreover, being a theorem in our logic, $A \rightarrow^{\mathrm{w}}_{\mathrm{d}} A$ is not a **CDLSN**-theorem. Well-known intuitionistically invalid formulae with (both strong and discussive) negations

[21] Their paracompleteness is in line with the history of logic, where paraconsistency and paracompleteness often go hand-in-hand. As J.-Y. Béziau puts it: "Paraconsistent logic and paracomplete logic appear therefore like husband and wife" [9, p. 12].

[22] The alternative approach which we sketch in Section 3 above is to interpret discussive negation in a non-standard way as $\neg_{\mathrm{d}} A =_{\mathrm{df}} \Box\neg A$. It gives us an **S5**-invalid formula $A \vee \Box\neg A$.

[23] Let us notice that in general, paracompleteness (when referring to the invalidity of the law of excluded middle) has not to entail that the implicational-negative part cannot behave classically (it can be easily justified by considering a similar translation to ours, where in the case of implication no modality is added). On the other hand, in $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$ for example, the formula $(\neg p \rightarrow^{\mathrm{w}}_{\mathrm{d}} \neg q) \rightarrow^{\mathrm{w}}_{\mathrm{d}} (q \rightarrow^{\mathrm{w}}_{\mathrm{d}} p)$ belonging to classical logic expressed in the implicational-negative language, is not a thesis of $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$. It can be invalidated by using our translations. Indeed, consider the formulas obtained *via* the translation σ given in Section 3 and equivalent on the basis of **S5** to each of the following formulas: $\Box(\Box\neg p \rightarrow \neg q) \rightarrow (\Box q \rightarrow p)$; $\Box(\Diamond p \vee \neg q) \rightarrow (\Box q \rightarrow p)$; $(\Diamond p \vee \Box\neg q) \rightarrow (\Box q \rightarrow p)$; $((\Diamond p \wedge \Box q) \vee (\Box\neg q \wedge \Box q)) \rightarrow p$. One can easily see that the last formula is not a thesis of **S5**, so, $(\neg p \rightarrow^{\mathrm{w}}_{\mathrm{d}} \neg q) \rightarrow^{\mathrm{w}}_{\mathrm{d}} (q \rightarrow^{\mathrm{w}}_{\mathrm{d}} p)$ is indeed not a thesis of $\mathbf{D}^{\mathsf{p}}_{\mathbf{2}}$.

predictably fail in **CDLSN**, say, $\neg_{\rm d} \neg_{\rm d} A \rightarrow^{\rm w}_{\rm d} A$, while their analogues with the classical negation, say, $\neg\neg A \rightarrow^{\rm w}_{\rm d} A$, are theorems in our logic. On the other hand, say, $A \rightarrow^{\rm w}_{\rm d} \neg_{\rm d} \neg_{\rm d} A$ as well as its analogue with the classical negation $A \rightarrow^{\rm w}_{\rm d} \neg\neg A$ are valid in **CDLSN** and in our logic, respectively. At last, in [2], the authors define discursive implication as material implication using discursive negation, i.e., $A \rightarrow^{\rm w}_{\rm d} B =_{\rm df} \neg_{\rm d} A \vee B$. Our analogue of this definition, $A \rightarrow^{\rm w}_{\rm d} B =_{\rm df} \neg A \vee_{\rm d} B$, as well as $\neg(A \wedge B) \rightarrow^{\rm w}_{\rm d} (\neg A \vee_{\rm d} \neg B)$ do not hold. Let us recall to the reader that our logic does not employ any discussive conjunction, for the motivation is not focused on it but on discussive disjunction.

7. Conclusion

With regard to future topics to study, let us point out two directions. The former deals with developing the target logic. Following Perzanowski's idea (which he introduced in a comment on his translation of Jaśkowski's paper [22, p. 59]), Ciuciura [11] considers a quasi-discursive system $\mathbf{ND_2^+}$ which has a discursive negation defined as follows:

- $\tau(\neg_d A) = \Diamond\neg\tau(A)$.

One may consider a paracomplete version of $\mathbf{ND_2^+}$ with the following negation:

- $\sigma(\neg_d A) = \Box\neg\sigma(A)$.

As the reviewer kindly drew our attention, it should be clear by looking at [37, Definition 11] that the three-valued logic $\mathbf{I^1}$ [47] is characterized in a similar manner by considering the translation as follows, where $\sim$ and $\rightarrow^I$ are negation and implication of $\mathbf{I^1}$:

- $\sigma({\sim}A) = \Box\neg\sigma(A)$,
- $\sigma(A \rightarrow^I B) = \Box\sigma(A) \rightarrow \Box\sigma(B)$.

The paper [37] also makes use of the 'diamond' type implication and, similarly to [11], the 'diamond-not' type negation in capturing $\mathbf{P^1}$ [46]:

- $\sigma({\sim}A) = \Diamond\neg\sigma(A)$,
- $\sigma(A \rightarrow^I B) = \Diamond\sigma(A) \rightarrow \Diamond\sigma(B)$.

Yet another similar translation can be found in Kovač's paper [28]:

- $\sigma(A \wedge B) = \Diamond\sigma(A) \wedge \Diamond\sigma(B)$,
- $\sigma(A \vee B) = \Box\sigma(A) \vee \Box\sigma(B)$,
- $\sigma(A \to B) = \Diamond\sigma(A) \to \Box\sigma(B)$.

Jaśkowski is known to have rejected a many-valued tabular approach to $\mathbf{D_2}$ with providing (arguably, quite weak) arguments in favour of the modal approach that has been fruitfully developing for more than six decades already (and the present paper is another one evidence of it). Nevertheless, as in the case of $\mathbf{D_2}$, it would be interesting to develop the tabular many-valued approach to $\mathbf{D_2^p}$: in particular, to find out $\mathbf{D_2^p}$-provabiity of (some of) the formulae that are characteristic of paracomplete reasoning.

The latter direction of future research deals with applications of $\mathbf{D_2^p}$. Generally, the reader's brief look at the axioms of $\mathbf{D_2^p}$ acknowledges their awkwardness: $\mathbf{D_2^p}$ inherits this property from $\mathbf{D_2}$. As a result, it is extremely difficult to implement the current Hilbert-style axiomatization of $\mathbf{D_2^p}$ in practice, which implies the problem with proof searching. A possible solution to this problem would be to axiomatize $\mathbf{D_2^p}$ as a Gentzen-style (sequent-style) or a natural deduction calculus. To the best of the authors' knowledge, no such calculi have yet been set up in the literature, not even for $\mathbf{D_2}$. We believe that on this path there will be found a solution for the notoriously difficult problem of independence of the axioms of discussive logics that is still open even for $\mathbf{D_2}$.

On the other hand, let us remind the reader about the passage on page 36 about $\mathbf{D_2^p}$ modeling a discussion whose debaters are not equal in the sense in which they are equal in a discussion modeled with $\mathbf{D_2}$. Such modeling, which is an application of the target logic to argumentation theory, would also stimulate setting up axiomatizations of $\mathbf{D_2^p}$ mentioned above.

Last, but not least, the present approach could be generalized by using other (weaker) modal logics as a basis for corresponding systems, similar to how the minimal variant $\mathbf{D_0}$ of $\mathbf{D_2}$ is axiomatized with the help of the deontic normal logic $\mathbf{D}$ in [19].

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