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Hamzeh Mohammadi

LINEAR ABELIAN MODAL LOGIC

Abstract

A many-valued modal logic, called linear abelian modal logic $\mathbf{LK}(\mathbf{A})$ is introduced as an extension of the abelian modal logic $\mathbf{K}(\mathbf{A})$. Abelian modal logic $\mathbf{K}(\mathbf{A})$ is the minimal modal extension of the logic of lattice-ordered abelian groups. The logic $\mathbf{LK}(\mathbf{A})$ is axiomatized by extending $\mathbf{K}(\mathbf{A})$ with the modal axiom schemas $\Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$ and $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$. Completeness theorem with respect to algebraic semantics and a hypersequent calculus admitting cut-elimination are established. Finally, the correspondence between hypersequent calculi and axiomatization is investigated.

Keywords: many-valued logic, modal logic, abelian logic, hypersequent calculus, cut-elimination.

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1. Introduction

Many-valued modal logics combine the Kripke frame semantics of classical modal logic with a many-valued semantics at each world. As in the classical setting, they provide a compromise between the good computational properties (decidability and low complexity) of propositional logics and the expressivity of first-order logics. Such logics have been used to model modal notions such as fuzzy similarity measures [14], fuzzy modal logic for belief functions (see, e.g., [13, 11]), probabilistic logics (see, e.g., [12, 21]), many-valued tense logics (see, e.g., [9, 16]), Łukasiewicz μ-calculus [22], continuous propositional modal logic [3], and serve as a basis for defining

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fuzzy description logics (see, e.g., [2, 15, 24]), dealing with fuzzy concepts and ontologies.

Several many-valued modal logics with propositional connectives interpreted in the ordered additive group of real numbers have been studied. These logics make use of basic operations on the real numbers and have been studied in a wide range of different contexts.

Recently, monadic logic of ordered abelian groups [19] and abelian modal logic $\mathbf{K}(\mathbf{A})$ [10] are introduced by G. Metcalfe and co-authors. Monadic logic of ordered abelian groups serves as a modal counterpart of the one-variable fragment of a (monadic) first-order real-valued logic. Propositional connectives are interpreted as the usual lattice and group operations over the real numbers in abelian modal logic $\mathbf{K}(\mathbf{A})$.

Abelian modal logic $\mathbf{K}(\mathbf{A})$ is the minimal modal extension of the abelian logic $\mathbf{A}$. Abelian logic $\mathbf{A}$ is the logic of lattice-ordered abelian groups, introduced independently by Meyer and Slaney [20] as a relevance logic, and Casari [4] as a comparative logic. In both settings, $\mathbf{A}$ was defined via axiom systems that are complete with respect to validity in the variety of lattice-ordered abelian groups.

As mentioned in [19], there are several advantages to focusing on modal extensions of Abelian logic, including that the language is rich enough to interpret other logics (e.g., modal extensions of Lukasiewicz logic), the semantics are based directly on structures studied in algebra and computer science, and the logics are naturally separated into the group and lattice fragments.

In [17], two embeddings of Łukasiewicz logic into Meyer and Slaney's Abelian logic and analytic proof systems for abelian logic are presented. In [10], a tableau calculus for the full logic $\mathbf{K}(\mathbf{A})$ and a sequent calculus for the modal-multiplicative fragment of $\mathbf{K}(\mathbf{A})$ as first steps towards addressing the corresponding (much more challenging) problems for the full logic, and complexity result are obtained.

The first main contribution of this work is to provide an axiomatization and algebraic semantics for the full logic $\mathbf{K}(\mathbf{A})$, which is addressed as an open question in the concluding remarks of [10]. The second aim is to develop a hypersequent calculus for the full logic $\mathbf{K}(\mathbf{A})$.

A real-valued modal logic, called linear abelian modal logic $\mathbf{LK}(\mathbf{A})$, as an extension of the minimal normal modal logic $\mathbf{K}(\mathbf{A})$ is introduced. An axiom system and also algebraic semantics for $\mathbf{LK}(\mathbf{A})$ are presented. Indeed, $\mathbf{LK}(\mathbf{A})$ is an extension of $\mathbf{K}(\mathbf{A})$ with the modal axiom schemas:

$\Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$ and $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$. The converse of the these axioms, i.e., $(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$ and $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$ is derivable in $\mathbf{LK}(\mathbf{A})$. Thus, the modal operator $\Box$ distributes over the both operators $\vee$ and $\wedge$ with an equivalence. It is well known that usually, necessity doesn't distribute over disjunction with an equivalence in the modal logic. So, it may be interesting to study logics like $\mathbf{LK}(\mathbf{A})$ in which necessity distributes over disjunction with an equivalence.

Moreover, completeness of the axiom system with respect to both corresponding appropriate algebras and linearly ordered algebras with a lattice-ordered abelian groups reduct, using methods of abstract algebraic logic is investigated. A hypersequent calculus called **HLK(A)** for $\mathbf{LK}(\mathbf{A})$, extending the sequent calculus for the modal-multiplicative fragment of $\mathbf{K}(\mathbf{A})$ (introduced in [10]) is presented. Finally, the cut-elimination theorem and the correspondence between the hypersequent calculus and the axiomatization are established.

The paper is structured as follows. In the next section, syntax and semantics of Linear Abelian Modal Logic are introduced. Then, in Section 3 the completeness theorem with respect to both appropriate algebras and linearly ordered algebras is proved. The cut-elimination theorem as well as the correspondence between the hypersequent calculus and the axiomatization are investigated in Section 4. Finally, Section 5 concludes the paper.

2. Linear abelian modal logic

In this section, we introduce a many-valued modal logic, namely linear abelian modal logic $\mathbf{LK}(\mathbf{A})$ as an extension of the minimal normal modal logic $\mathbf{K}(\mathbf{A})$ extending Abelian logic $\mathbf{A}$, the logic of lattice-ordered abelian groups. We provide an axiom system and also algebraic and Kripke semantics for $\mathbf{LK}(\mathbf{A})$. Finally, we establish a connection between algebraic and Kripke semantics.

2.1. Axiomatizations

The language $\mathcal{L}_A^{\Box}$ of linear abelian modal logic $\mathbf{LK}(\mathbf{A})$ is consisting of the binary connective $\wedge, \vee, \to$ and unary connective $\Box$. The formula of $\mathbf{LK}(\mathbf{A})$ is defined inductively by

$$\varphi := p \,|\, \varphi \wedge \psi \,|\, \varphi \vee \psi \,|\, \varphi \to \psi \,|\, \Box\varphi,$$

where p is a propositional variable. To define further connectives, let

$$\bar{0} := p \to p, \quad \neg\varphi := \varphi \to \bar{0}, \quad \varphi + \psi := \neg\varphi \to \psi, \quad \Diamond\varphi := \neg\Box\neg\varphi,$$

and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$. We also define $0\varphi := \bar{0}$ and $(n+1)\varphi := \varphi + (n\varphi)$ for each $n \in \mathbb{N}$. Let us also denote by Fm the set of formulas of $\mathcal{L}_A^{\Box}$ over a countably infinite set of variables. An axiomatization of the minimal normal modal logic $\mathbf{K}(\mathbf{A})$ is presented in Table 1. An axiom system of

Table 1. An Axiom System for Abelian Modal Logic $\mathbf{K}(\mathbf{A})$

(B)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(I)	$\varphi \to \varphi$
(C)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(A)	$((\varphi \to \psi) \to \psi) \to \varphi$
(+1)	$\varphi \to (\psi \to \varphi + \psi)$
(+2)	$(\varphi \to (\psi \to \chi)) \to ((\varphi + \psi) \to \chi)$
($\bar{0}$1)	$\bar{0}$
($\bar{0}$2)	$\varphi \to (\bar{0} \to \varphi)$
($\wedge$1)	$(\varphi \wedge \psi) \to \varphi$
($\wedge$2)	$(\varphi \wedge \psi) \to \psi$
($\wedge$3)	$((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$
($\vee$1)	$\varphi \to (\varphi \vee \psi)$
($\vee$2)	$\psi \to (\varphi \vee \psi)$
($\vee$3)	$((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$
(K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
($\mathrm{D_n}$)	$\Box(n\varphi) \to n\Box\varphi \quad (n \geq 2)$
	$\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$ (mp) $\qquad \dfrac{\varphi}{\Box\varphi}$ (nec) $\qquad \dfrac{\varphi \qquad \psi}{\varphi \wedge \psi}$ (adj)

linear abelian modal logic $\mathbf{LK}(\mathbf{A})$ is defined over $\mathcal{L}_A^{\Box}$ by extending $\mathbf{K}(\mathbf{A})$ with the following modal axiom schemas:

$$(\vee_{\Box}) \quad \Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi),$$

$$(\wedge_{\Box}) \quad (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi).$$

For a formula $\varphi \in \mathrm{Fm}$, we write $\vdash_{\mathbf{LK(A)}} \varphi$ if there exists a $\mathbf{LK}(\mathbf{A})$-derivation of φ, defined as usual as a finite sequence of $\mathcal{L}_A^{\Box}$-formulas that ends with φ and is constructed inductively using the axioms and rules of $\mathbf{LK}(\mathbf{A})$.

PROPOSITION 2.1. For any $\varphi, \psi \in \mathrm{Fm}$,

(i) $\vdash_{\mathbf{LK(A)}} (\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$,

(ii) $\vdash_{\mathbf{LK(A)}} \Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$,

(iii) $\vdash_{\mathbf{LK(A)}} n\Box\varphi \to \Box(n\varphi) \quad (n \geq 2)$.

PROOF: Derivation for (i) is obtained, using the axiom schemas $(\vee 1), (\mathrm{K})$, and $(\vee 3)$, and also rules $(\mathrm{nec}), (\mathrm{mp})$ and (adj) as follows:

1. $\vdash_{\mathbf{LK(A)}} \varphi \to (\varphi \vee \psi) \qquad (\vee 1)$
2. $\vdash_{\mathbf{LK(A)}} \Box(\varphi \to (\varphi \vee \psi)) \qquad (\mathrm{nec})$
3. $\vdash_{\mathbf{LK(A)}} \Box(\varphi \to (\varphi \vee \psi)) \to (\Box\varphi \to \Box(\varphi \vee \psi)) \qquad (\mathrm{K})$
4. $\vdash_{\mathbf{LK(A)}} \Box\varphi \to \Box(\varphi \vee \psi) \qquad (2, 3 \text{ and } (mp))$
5. $\vdash_{\mathbf{LK(A)}} \Box\psi \to \Box(\varphi \vee \psi) \qquad (\text{similarly})$
6. $\vdash_{\mathbf{LK(A)}} (\Box\varphi \to \Box(\varphi \vee \psi)) \wedge (\Box\psi \to \Box(\varphi \vee \psi)) \qquad (4, 5 \text{ and } (\mathrm{adj}))$
7. $\vdash_{\mathbf{LK(A)}} (\Box\varphi \to \Box(\varphi \vee \psi)) \wedge (\Box\psi \to \Box(\varphi \vee \psi)) \to ((\Box\varphi \vee \Box\psi) \to (\Box(\varphi \vee \psi))) \qquad (\vee 3)$
8. $\vdash_{\mathbf{LK(A)}} (\Box\varphi \vee \Box\psi) \to (\Box(\varphi \vee \psi)) \qquad (6, 7 \text{ and } (\mathrm{mp})$

Derivation for (ii) is obtained, similar to the derivation of (i), using the axiom schemas $(\wedge 1), (\mathrm{K})$ and $(\wedge 3)$, and also rules $(\mathrm{nec}), (\mathrm{mp})$ and (adj), and is omitted here. For derivation of (iii), observe that $n\Box\varphi \to \Box(n\varphi)$ is derivable in $\mathbf{LK(A)}$ for $n \geq 2$ using (nec) and (mp) together with the axioms of $\mathbf{LK(A)}$. For instance, $(\Box\varphi + \Box\varphi) \to \Box(\varphi + \varphi)$ is derivable as follows:

1. $\vdash_{\mathbf{LK(A)}} \varphi \to (\varphi \to (\varphi + \varphi)) \qquad (+1)$
2. $\vdash_{\mathbf{LK(A)}} \Box(\varphi \to (\varphi \to (\varphi + \varphi))) \qquad (\mathrm{nec})$
3. $\vdash_{\mathbf{LK(A)}} \Box(\varphi \to (\varphi \to (\varphi + \varphi))) \to (\Box\varphi \to \Box(\varphi \to (\varphi + \varphi))) \qquad (\mathrm{K})$

4. $\vdash_{\mathbf{LK(A)}} \Box\varphi \to \Box(\varphi \to (\varphi + \varphi))$ (2, 3 and (mp))
5. $\vdash_{\mathbf{LK(A)}} \Box(\varphi \to (\varphi + \varphi)) \to (\Box\varphi \to \Box(\varphi + \varphi))$ (K)
6. $\vdash_{\mathbf{LK(A)}} \Box\varphi \to (\Box\varphi \to \Box(\varphi + \varphi))$ ((B), 4, 5 and (mp))
7. $\vdash_{\mathbf{LK(A)}} (\Box\varphi \to (\Box\varphi \to \Box(\varphi{+}\varphi))) \to (\Box\varphi{+}\Box\varphi \to \Box(\varphi{+}\varphi))$ (+2)
8. $\vdash_{\mathbf{LK(A)}} \Box\varphi + \Box\varphi \to \Box(\varphi + \varphi)$ (6, 7 and (mp)) □

2.2. Semantics

In this subsection, algebraic semantics for **LK**(**A**) are presented. Appropriate class of algebras for **LK(A)** is defined over lattice-ordered abelian groups.

DEFINITION 2.2. A lattice-ordered abelian group (abelian ℓ-group for short) is an algebraic structure $(A, \wedge, \vee, +, \neg, \bar{0})$ such that $(A, +, \neg, \bar{0})$ is an abelian group, $(A, \wedge, \vee)$ is a lattice, and $a+(b\vee c) = (a+b)\vee(a+c)$ for all $a, b, c \in A$. In addition, we define $a \to b = \neg a + b$, and $a \leq b$ iff $a \vee b = b$.

Well-known examples of abelian ℓ-groups are

$$\text{the integers } \mathcal{Z} = (\mathbb{Z}, \min, \max, +, -, 0),$$
$$\text{the rationals } \mathcal{Q} = (\mathbb{Q}, \min, \max, +, -, 0),$$
$$\text{and the reals } \mathcal{R} = (\mathbb{R}, \min, \max, +, -, 0).$$

In fact, any of them generates the variety of Abelian ℓ-groups (see [18] for more details).

Below we introduce algebras for the logic defined in the previous section, the idea being to consider particular classes of residuated lattices where the modal operator is interpreted by a special unary operator I on the corresponding algebras.

DEFINITION 2.3 (**LK(A)**-algebra). An **LK(A)**-algebra is an algebra $\mathcal{A} = (A, \wedge, \vee, +, \neg, \bar{0}, I)$, where the reduct $(A, \wedge, \vee, +, \neg, \bar{0})$ is an abelian ℓ-group and I is an unary operation satisfying:

1. $I(x \to y) \leq I(x) \to I(y)$,
2. $I(x \vee y) = I(x) \vee I(y)$,
3. $I(x \wedge y) = I(x) \wedge I(y)$,

4. $I(x+x) = I(x) + I(x)$,
5. $I(\bar{0}) = \bar{0}$.

An $\mathcal{A}$-valuation is a function $V : \mathrm{Fm} \to A$ satisfying $V(\varphi \star \psi) = V(\varphi) \star V(\psi)$ for $\star \in \{\wedge, \vee, \to, +\}$, and $V(\Box\varphi) = I(V(\varphi))$. Formula φ is $\mathcal{A}$-valid if $V(\varphi) \geq \bar{0}$ for each $\mathcal{A}$-valuation V. We write $\models_{\mathbf{LK(A)}} \varphi$ iff φ is valid in all $\mathbf{LK(A)}$-algebras.

Example 2.4. Consider the real number structure $\mathcal{R} = (\mathbb{R}, \min, \max, +, -, 0, I)$, where I is defined as follows:

$$I : \mathbb{R} \longrightarrow \mathbb{R}$$
$$I(x) = \min\{x, 0\},$$

One can easily prove that this structure is an $\mathbf{LK(A)}$-algebra. Note that $\min\{x+y, 0\} \neq \min\{x, 0\} + \min\{y, 0\}$ (consider, for example $x = 1$ and $y = -1$), i.e., $I(x+y) \neq I(x) + I(y)$. While, $\min\{x+x, 0\} = \min\{x, 0\} + \min\{x, 0\}$, i.e., $I(x+x) = I(x) + I(x)$.

3. Completeness

In this section, we will establish the completeness theorem with respect to the corresponding algebraic semantics proceeding in the standard way (see e.g [18, 5, 8]). Given $T \subseteq \mathrm{Fm}$, the Lindenbaum algebra is defined in the usual way as $\mathcal{A}_T = (A_T, \wedge_T, \vee_T, +_T, \neg_T, \bar{0}_T, I_T)$ where $A_T = \{[\varphi]_T : \varphi \in \mathrm{Fm}\}$, $[\varphi]_T = \{\psi \in \mathrm{Fm} : T \vdash_{\mathbf{LK(A)}} \varphi \leftrightarrow \psi\}$, $[\varphi]_T \star_T [\psi]_T = [\varphi \star \psi]_T$ for $\star \in \{+, \vee, \wedge\}$, $\neg_T[\varphi] = [\neg\varphi]_T$, $\bar{0}_T = [\bar{0}]_T$, and $I_T[\varphi]_T = [\Box\varphi]_T$. The next Lemma follows from various provabilities in $\mathbf{LK}(\mathbf{A})$ and the axioms.

Lemma 3.1. *$\mathcal{A}_T$ is an $\mathbf{LK(A)}$-algebra.*

To show that $\mathcal{A}_T$-validity corresponds to $\mathbf{LK}(\mathbf{A})$-derivability from T, we make use of a specially defined valuation for this algebra that maps each formula to its corresponding equivalence class.

Lemma 3.2. *For any $T \subseteq \mathrm{Fm}$ and $\varphi \in \mathrm{Fm}$:*

$$T \vdash_{\mathbf{LK(A)}} \varphi \quad \text{iff} \quad \bar{0} \leq V_T(\varphi),$$

where V_T is the $\mathcal{M}_T$-valuation defined by $V_T(p) = [p]_T$ for each propositional variable p.

Proof: We first prove that $V_T(\varphi) = [\varphi]_T$ for all formulas φ, by induction on the complexity of φ. The case where φ is a variable follows by definition. For the other cases, just note that for any connective $\star \in \{+, \vee, \wedge\}$ (using the induction hypothesis):

$$\begin{aligned} V_T(\varphi \star \psi) &= V_T(\varphi) \star V_T(\psi) \\ &= [\varphi]_T \star [\psi]_T \\ &= [\varphi \star \psi]_T \end{aligned}$$

For unary connective $\Box$, we have: $V_T(\Box\varphi) = I_T(V_T(\varphi)) = I_T([\varphi]_T) = [\Box\varphi]_T$. The result then follows because $[\bar{0}]_T \leq [\varphi]_T$ iff $T \vdash_{\mathbf{LK(A)}} \bar{0} \to \varphi$ iff $T \vdash_{\mathbf{LK(A)}} \varphi$. $\Box$

Theorem 3.3 (Completeness). *$T \models_{\mathbf{LK(A)}} \varphi$ iff $T \vdash_{\mathbf{LK(A)}} \varphi$.*

Proof: Soundness proceeds as usual by an induction on the height of a derivation of φ in $\mathbf{LK}(\mathbf{A})$, showing that each axiom is valid and each rule sound in all $\mathbf{LK(A)}$-algebras. For the reverse direction, assume that $T \nvdash_{\mathbf{LK(A)}} \varphi$. By the previous lemma, $V_T(\psi) \geq \bar{0}$ for each $\psi \in T$ where $V_T(\varphi) \ngeq \bar{0}$. So $T \nvDash \varphi$. $\Box$

We now turn our attention, following [6, 7, 8], next to completeness with respect to linearly ordered algebras. First, let us say that a congruence filter of an $\mathbf{LK(A)}$-algebra $\mathcal{A}$ is a set $F = \{x \in A : \exists y \leq x \; (y\theta\bar{0})\}$, for some congruence θ on $\mathcal{A}$. The next Lemma follows from the fact that the reduct of an $\mathbf{LK(A)}$-algebra is an abelian ℓ-group.

Lemma 3.4. *Let $\mathcal{A} = (A, \wedge, \vee, +, \neg, \bar{0}, I)$ be an $\mathbf{LK(A)}$-algebra and $a, b, c, d \in A$. If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.*

Corollary 3.5. Let $\mathcal{A} = (A, \wedge, \vee, +, \neg, \bar{0}, I)$ be an $\mathbf{LK(A)}$-algebra and $a, b \in A$. If $a, b \leq \bar{0}$, then $(a + b) \leq (a \vee b)$.

Proof: Let $a, b \leq \bar{0}$, then $a \vee b \leq \bar{0}$ and so, by Lemma 3.4, $(a \vee b) + (a \vee b) \leq a \vee b$ since $(a \vee b) \leq (a \vee b)$. Now, $a \leq a \vee b$ and $b \leq a \vee b$ follows that $a + b \leq (a \vee b) + (a \vee b) \leq a \vee b$. $\Box$

Lemma 3.6. *F is a congruence filter of an $\mathbf{LK(A)}$-algebra $\mathcal{A}$ iff (i) $\bar{0} \in F$ (ii) if $a \in F$ and $a \to b \in F$, then $b \in F$ (iii) if $a \in F$ then $I(a) \in F$.*

Proof: That a congruence filter must satisfy (i), is almost immediate. We check (ii) and (iii). If $a \in F$ and $a \to b \in F$, then there are $u, v \in A$

such that $u \leq a$, $v \leq a \to b$ and $u\theta a$ and $v\theta(a \to b)$. So, by Lemma 3.4, $u + v \leq a + (a \to b)$ i.e., $u + v \leq a + (\neg a + b)$. Therefore, by equations $\bar{0} + a = a$ and $\neg a + a = \bar{0}$ of the definition of abelian ℓ-group and the compatibility of congruence with $(+)$, we have $(u + v) \leq b$ and $(u + v)\theta b$. Thus, $b \in F$. If $a \in F$, then there is $u \in A$ such that $u \leq a$ and $u\theta a$. It follows that $I(u) \leq I(a)$ and $I(u)\theta I(a)$, since $u \leq a$ i.e., $u \vee a = a$ follows that $I(u \vee a) = I(a)$ so $I(u) \vee I(a) = I(a)$ i.e., $I(u) \leq I(a)$, and hence $I(a) \in F$. Conversely, let F be a subset of A that satisfies the conditions, and let θ be defined by $a\theta b$ iff $a \to b \in F$ and $b \to a \in F$. One can easily show that θ is a equivalence relation. Thus, we may define equivalence classes $[a]_F = \{b \mid a\theta b\}$. We prove that θ is compatible with the operations of **LK(A)**-algebras.

- θ is compatible with $(+)$: If $a\theta b$ and $c\theta d$, then $a \to b, b \to a \in F$ and $c \to d, d \to c \in F$, therefore $(a \to b)+(c \to d), (b \to a)+(d \to c) \in F$, as F is closed under $(+)$. It follows that $(\neg a + b) + (\neg c + d), (\neg b + a) + (\neg d + c) \in F$, and so $\neg(a + c) + (b + d), \neg(b + d) + (a + c) \in F$ i.e., $(a + c) \to (b + d), (b + d) \to (a + c) \in F$. Thus, $(a + c)\theta(b + d)$.

- θ is compatible with $(\vee)$: Since θ is an equivalence relation, we define equivalence classes $[a]_\theta = \{b \mid a\theta b\}$. Let A/θ_F be the set of all equivalence classes. One verifies that $(A/\theta_F, \cap, \cup, +_F, \neg_F, 0_F, I_F)$, where $\cap, \cup, +_F, \neg_F, 0_F, I_F$ are defined component-wise from the ones of $\mathcal{A}$, is an **LK(A)**-algebra. If $a\theta b$ and $c\theta d$, then $[a]_\theta = [b]_\theta$ and $[c]_\theta = [d]_\theta$. It follows that $[a]_\theta \cup [c]_\theta = [b]_\theta \cup [d]_\theta$, and so $[a \vee c]_\theta = [b \vee d]_\theta$. Therefore, $(a \vee c)\theta(b \vee d)$. The compatibility of θ with $(\wedge)$ is treated similarly.

- θ is compatible with $(\neg)$: If $a\theta b$, then $a \to b, b \to a \in F$, i.e., $\neg a + b, \neg b + a \in F$. Therefore, $\neg b + \neg(\neg a), \neg a + \neg(\neg b) \in F$, i.e., $\neg a \to \neg b, \neg b \to \neg a \in F$. Thus $\neg a\theta\neg b$.

- θ is compatible with (I): If $a\theta b$, then $a \to b, b \to a \in F$. Therefore $I(a \to b), I(b \to a) \in F$, as F is closed under I. It follows that $I(a) \to I(b), I(b) \to I(a) \in F$. Thus, $I(a)\theta I(b)$. □

Now, by imitating [6], we define $\mathrm{Fg}(a)$ be the smallest congruence filter containing a, and define inductively: $I_0(a) = a$ and $I_{n+1}(a) = I(I_n(a)) \wedge I_n(a)$ for an **LK(A)**-algebra $\mathcal{A}$ and $a \in A$. Note that $I_{n+1}(a) \leq I_n(a)$, thus, by induction, $I_n(a) \leq I_m(a)$ for $m \leq n$.

Lemma 3.7. *Let $\mathcal{A} = (A, \wedge, \vee, +, \neg, \bar{0}, I)$ be an* **LK(A)***-algebra and $a, b \in A$. If $a \leq b$, then $I_n(a) \leq I_n(b)$ for all $n \in \mathbb{N}$.*

Proof: We first observe that $a \leq b$ if and only if $I(a) \leq I(b)$:

$$a \leq b \,\text{iff}\, a \vee b = b \,\text{iff}\, I(a \vee b) = I(b) \,\text{iff}\, I(a) \vee I(b) = I(b) \,\text{iff}\, I(a) \leq I(b).$$

Let $a \leq b$, by induction on n we can easily prove $I_n(a) \leq I_n(b)$. For $n = 0$, obviously $I_0(a) \leq I_0(b)$. Suppose $I_n(a) \leq I_n(b)$, then $I(I_n(a)) \leq I(I_n(b))$. It follows that $I(I_n(a)) \wedge I_n(a) \leq I(I_n(b)) \wedge I_n(b)$ i.e., $I_{n+1}(a) \leq I_{n+1}(b)$. □

Lemma 3.8. *Let $\mathcal{A} = (A, \wedge, \vee, +, \neg, \bar{0}, I)$ be an* **LK(A)***-algebra and $a, b \in A$. Then $I_n(a \vee b) = I_n(a) \vee I_n(b)$ for all $n \in \mathbb{N}$.*

Proof: First observe that by induction on n we can easily prove $I_n(a) \leq a$ for all $n \in \mathbb{N}$: For $n = 0$, $I_0(a) = a \leq a$. Suppose $I_n(a) \leq a$, then $I_{n+1}(a) = I(I_n(a)) \wedge I_n(a) \leq I_n(a) \leq a$. Suppose now that $I_n(a \vee b) = I_n(a) \vee I_n(b)$, then $I(I_n(a \vee b)) = I(I_n(a) \vee I_n(b))$, so $I(I_n(a \vee b)) = I(I_n(a)) \vee I(I_n(b))$. It follows that $I(I_n(a \vee b)) \wedge I_n(a \vee b) = (I(I_n(a)) \wedge I_n(a)) \vee (I(I_n(b)) \wedge I_n(b))$ i.e., $I_{n+1}(a \vee b) = I_{n+1}(a) \vee I_{n+1}(b)$. □

Lemma 3.9. *Let $\mathcal{A} = (A, \wedge, \vee, +, \neg, \bar{0}, I)$ be an* **LK(A)***-algebra and $a \in A$. Then*

$$\mathrm{Fg}(a) = \{x \in A \mid \exists n, m \in \mathbb{N}\,(mI_n(a) \leq x)\},$$

where $1I_n(a) = I_n(a)$ and $(n+1)I_n(a) = I_n(a) + nI_n(a)$.

Proof: Let $G = \{x \in A \mid \exists n, m \in \mathbb{N}\,(mI_n(a) \leq x)\}$. We show that $G \subseteq \mathrm{Fg}(a)$; suppose $x \in G$, then there is $n, m \in \mathbb{N}$ such that $mI_n(a) \leq x$. It follows that $x \in \mathrm{Fg}(a)$ because $a \in \mathrm{Fg}(a)$ and $\mathrm{Fg}(a)$ is closed upwards and closed under $I, +$, and $\wedge$. For the opposite direction, since $a \in G$, it suffices to prove that G is a congruence filter. It is trivial that $\bar{0} \in G$. If $x, x \to y \in G$, then there are $m_1, n_1, m_2, n_2 \in \mathbb{N}$ such that $m_1(I_{n_1}(a)) \leq x$ and $m_2(I_{n_2}(a)) \leq x \to y$. But then easily $(m_1+m_2)(I_{n_1+n_2}(a)) \leq x+(x \to y) = x + (\neg x + y) = y$, and hence $y \in G$. Finally, G is closed under I. If $x \in G$, then there are an m, n such that $m(I_n(a)) \leq x$. It follows that $mI_{n+1}(a) \leq mI(I_n(a)) = I(mI_n(a)) \leq I(x)$, and $I(x) \in G$. Thus, by Lemma 3.6, G is a filter and $a \in G$. It follows that $\mathrm{Fg}(a) \subseteq G$. □

Theorem 3.10. *Every subdirectly irreducible* **LK(A)***-algebra $\mathcal{A}$ is linearly ordered.*

Proof: Assume for a contradiction that $\mathcal{A}$ is a subdirectly irreducible **LK(A)**-algebra with minimum non-trivial filter F and elements a, b such that $a \not\leq b$ and $b \not\leq a$. Then, both $\mathrm{Fg}(a \to b)$ and $\mathrm{Fg}(b \to a)$ are non-trivial filters; hence they both contain F. Let $c \in F$ with $c < \bar{0}$. Then, there are $m_1, n_1, m_2, n_2 \in \mathbb{N}$ such that $I_{n_1}(m_1(a \to b)) = m_1 I_{n_1}(a \to b) \leq c < \bar{0}$ and $I_{n_2}(m_2(b \to a)) = m_2 I_{n_2}(b \to a) \leq c < \bar{0}$. It follows, by Lemma 3.7, that $m_1(a \to b) < \bar{0}$ and $m_2(b \to a) < \bar{0}$. Let $m = \max\{m_1, m_2\}$, then $m(a \to b) < \bar{0}$ and $m(b \to a) < \bar{0}$. Therefore, by Lemma 3.5, $m(a \to b) + m(b \to a) \leq m(a \to b) \vee m(b \to a)$. Then, again by Lemma 3.7, $I_n(m(a \to b) + m(b \to a)) \leq I_n(m(a \to b) \vee m(b \to a))$ for all n. Now, letting $n = \max\{n_1, n_2\}$, we have the following contradiction:

$$\begin{aligned}
\bar{0} = I_n(\bar{0}) &= I_n((m(\neg a) + mb) + (m(\neg b) + ma)) \\
&= I_n(m(a \to b) + m(b \to a)) \\
&\leq I_n(m(a \to b) \vee m(b \to a)) \\
&= I_n(m(a \to b)) \vee I_n(m(b \to a)) \\
&= mI_n(a \to b) \vee mI_n(b \to a) \\
&\leq m_1 I_n(a \to b) \vee m_2 I_n(b \to a) \\
&\leq m_1 I_{n_1}(a \to b) \vee m_2 I_{n_2}(b \to a) \\
&\leq c \vee c = c < \bar{0}.
\end{aligned}$$

□

Hence, making use of Birkhoff's subdirect representation theorem, we have the following Corollary.

Corollary 3.11. Every **LK(A)**-algebra is isomorphic to a subdirect product of a family of linearly ordered **LK(A)**-algebras.

4. A hypersequent calculus for LK(A)

In this section, a proof system for **LK(A)**, called **HLK(A)** in the framework of hypersequent, is presented. Hypersequent is a generalization of sequents introduced independently by Avron [1] and Pottinger [23]. **HLK(A)** extends the sequent calculus for the modal multiplicative fragment of **K**(**A**) [10]. Then, the cut elimination theorem is established and finally it is shown that the axiomatic and hypersequent presentations really characterize the same logics.

Since in this section we will often be dealing with quite complicated structures, let us recall some notational conveniences:

- φ, ψ, χ and $\Gamma, \Delta, \Pi, \Sigma$ (sometimes with primes or numerical subscripts) denote arbitrary formulas and finite multisets of formulas, respectively. The multiset union $\Gamma \uplus \Delta$ is often denoted by Γ, Δ. In addition, $n\Gamma$ or sometimes Γ^n is used for $\Gamma, \ldots, \Gamma$ (n times), and $\Box\Gamma$ for $\{\Box\varphi : \varphi \in \Gamma\}$.

- a sequent is an ordered pair of finite multisets of formulas Γ and Δ, written $\Gamma \Rightarrow \Delta$. A hypersequent is a finite multiset of ordinary sequents, written $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$.

- $G, H, \mathcal{G}, \mathcal{H}$ (possibly with primes) denote hypersequents, $[\mathcal{G}_i]_{i=1}^n$ denotes the hypersequent $\mathcal{G}_1 \mid \ldots \mid \mathcal{G}_n$, and also $\{\mathcal{G}_i\}_{i=1}^n$ denotes a set of hypersequents $\mathcal{G}_1, \ldots, \mathcal{G}_n$ (perhaps the premises of some rule application).

The intended interpretation of the hypersequent $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ is defined as follows:

$$\mathcal{I}(H) = (\sum \Gamma_1 \to \sum \Delta_1) \vee \cdots \vee (\sum \Gamma_n \to \sum \Delta_n),$$

where $\Sigma\{\varphi_1, \ldots, \varphi_m\} := \varphi_1 + \ldots + \varphi_m$ and $\Sigma\emptyset = \bar{0}$. Axioms and rules of hypersequent calculus **HLK(A)** is presented in Table 2. For a hypersequent H, we write $\vdash_{\mathbf{HLK(A)}} H$ if there is a **HLK**(**A**)-derivation of H.
The following rules for other connectives are **HLK(A)**-derivable:

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta \mid H}{\Gamma, \varphi + \psi \Rightarrow \Delta \mid H}\,(L+) \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta \mid H}{\Gamma \Rightarrow \varphi + \psi, \Delta \mid H}\,(R+)$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \mid H}{\Gamma, \neg\varphi \Rightarrow \Delta \mid H}\,(L\neg) \qquad \frac{\Gamma, \varphi \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \neg\varphi, \Delta \mid H}\,(R\neg)$$

$$\frac{\Gamma \Rightarrow \Delta \mid H}{\Gamma, \bar{0} \Rightarrow \Delta \mid H}\,(L\bar{0}) \qquad \frac{\Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \bar{0}, \Delta \mid H}\,(R\bar{0})$$

Example 4.1. Below we provide an example of a **HLK(A)**-derivation to get more familiar with this calculus.

Table 2. Hypersequent Calculus **HLK(A)**

Axiom:

$$\dfrac{}{\Gamma \Rightarrow \Gamma \mid H}\ (\mathrm{AX})$$

Logical rules:

$$\dfrac{\Gamma, \psi \Rightarrow \varphi, \Delta \mid H}{\Gamma, \varphi \to \psi \Rightarrow \Delta \mid H}\ (L \to) \qquad \dfrac{\Gamma, \varphi \Rightarrow \psi, \Delta \mid H}{\Gamma \Rightarrow \Delta, \varphi \to \psi \mid H}\ (R \to)$$

$$\dfrac{\Gamma, \varphi \Rightarrow \Delta \mid \Gamma, \psi \Rightarrow \Delta \mid H}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta \mid H}\ (L\wedge) \qquad \dfrac{\Gamma \Rightarrow \varphi, \Delta \mid H \qquad \Gamma \Rightarrow \psi, \Delta \mid H}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi \mid H}\ (R\wedge)$$

$$\dfrac{\Gamma, \varphi \Rightarrow \Delta \mid H \qquad \Gamma, \psi \Rightarrow \Delta \mid H}{\Gamma, \varphi \vee \psi \Rightarrow \Delta \mid H}\ (L\vee) \qquad \dfrac{\Gamma \Rightarrow \varphi, \Delta \mid \Gamma \Rightarrow \psi, \Delta \mid H}{\Gamma \Rightarrow \Delta, \varphi \vee \psi \mid H}\ (R\vee)$$

Modal rule:

$$\dfrac{\Gamma \Rightarrow n\varphi \mid H}{\Box\Gamma \Rightarrow n\Box\varphi \mid H}\ (\Box_n)$$

Structural rules:

$$\dfrac{\Gamma, \varphi \Rightarrow \Delta \mid H \qquad \Pi \Rightarrow \varphi, \Sigma \mid H}{\Gamma, \Pi \Rightarrow \Sigma, \Delta \mid H}\ (\mathrm{Cut}) \qquad \dfrac{\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta \mid H}\ (\mathrm{EC})$$

$$\dfrac{\Gamma \Rightarrow \Delta \mid H \qquad \Pi \Rightarrow \Sigma \mid H}{\Gamma, \Pi \Rightarrow \Delta, \Sigma \mid H}\ (\mathrm{Mix}) \qquad \dfrac{\Gamma, \Pi \Rightarrow \Sigma, \Delta \mid H}{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma \mid H}\ (\mathrm{Split})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi \mid \psi \Rightarrow \varphi \wedge \psi}\ (\mathrm{AX}) \qquad \dfrac{\dfrac{\dfrac{}{\varphi, \psi \Rightarrow \varphi, \psi}\ (\mathrm{AX})}{\varphi \Rightarrow \psi \mid \psi \Rightarrow \varphi}\ (\mathrm{Split}) \qquad \dfrac{}{\varphi \Rightarrow \psi \mid \psi \Rightarrow \psi}\ (\mathrm{AX})}{\varphi \Rightarrow \psi \mid \psi \Rightarrow \varphi \wedge \psi}\ (R\wedge)}{\varphi \Rightarrow \varphi \wedge \psi \mid \psi \Rightarrow \varphi \wedge \psi}\ (R\wedge)}{\varphi \Rightarrow \varphi \wedge \psi \mid \Box\psi \Rightarrow \Box(\varphi \wedge \psi)}\ (\Box_1)}{\Box\varphi \Rightarrow \Box(\varphi \wedge \psi) \mid \Box\psi \Rightarrow \Box(\varphi \wedge \psi)}\ (\Box_1)}{\Box\varphi \wedge \Box\psi \Rightarrow \Box(\varphi \wedge \psi)}\ (L\wedge)}{\Rightarrow \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)}\ (R \to)$$

We now consider a more complicated family of rules, indexed by $k \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N}$, that is inspired by Denisa Diaconescu et al [10] and will be very useful in subsequent cut-elimination and completeness proofs:

$$\frac{\Gamma_0 \Rightarrow \;\mid H \quad \Gamma_1 \Rightarrow k\varphi_1 \mid H \quad \cdots \quad \Gamma_n \Rightarrow k\varphi_n \mid H}{\Delta, \Box\Gamma \Rightarrow \Box\varphi_1, \ldots, \Box\varphi_n, \Delta \mid H}\,(\Box_{k,n}) \quad \text{where } k\Gamma = \Gamma_0, \ldots, \Gamma_n$$

Critically for our later considerations, $\Box_{k,n}$ is **HLK(A)**-derivable for all $k \in \mathbb{N} \setminus \{0\}, n \in \mathbb{N}$ (for $k = 1$, omitting the applications of (EC) and (Split)):

$$\dfrac{\dfrac{}{\Delta \Rightarrow \Delta \mid H}\,(\text{AX}) \qquad \dfrac{\dfrac{\dfrac{\Gamma_0 \Rightarrow \;\mid H}{\Box\Gamma_0 \Rightarrow \;\mid H}\,(\Box_0) \quad \dfrac{\dfrac{\Gamma_1 \Rightarrow k\varphi_1 \mid H}{\Box\Gamma_1 \Rightarrow k\Box\varphi_1 \mid H}\,(\Box_k) \quad \begin{matrix}\dfrac{\dfrac{\Gamma_n \Rightarrow k\varphi_n \mid H}{\Box\Gamma_n \Rightarrow k\Box\varphi_n \mid H}\,(\Box_k)}{}\,(\text{Mix}) \\ \vdots\end{matrix}}{\Box(\Gamma_1 \ldots, \Gamma_n) \Rightarrow k\Box\varphi_1, \ldots, k\Box\varphi_n \mid H}\,(\text{Mix})}{\Box(\Gamma_0, \Gamma_1 \ldots, \Gamma_n) \Rightarrow k\Box\varphi_1, \ldots, k\Box\varphi_n \mid H}\,(\text{Mix})}{\Box\Gamma \Rightarrow \Box\varphi_1, \ldots, \Box\varphi_n \mid H}\,(\text{Split}), (\text{EC})}{\Delta, \Box\Gamma \Rightarrow \Box\varphi_1, \ldots, \Box\varphi_n, \Delta \mid H}\,(\text{Mix})$$

In order to prove the cut elimination theorem, we begin by showing that every cut-free **HLK(A)**-derivation can be transformed into a derivation in a restricted calculus $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$ consisting only of the rules (AX), logical rules, $(\Box_{k,n})(k \in \mathbb{N} \setminus \{0\}, n \in \mathbb{N})$, (Split) and (EC).

LEMMA 4.2. *The following rules are height-preserving* $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$*-admissible.*

$$\frac{H}{\Gamma \Rightarrow \Delta \mid H}\,(\text{EW}) \qquad \frac{\Gamma \Rightarrow \Delta \mid H}{\Gamma, \Pi \Rightarrow \Delta, \Pi \mid H}\,(\text{IW})$$

PROOF: By induction on the height of the premises. □

LEMMA 4.3. *All logical rules are* $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$*-invertible.*

PROOF: To cope with multiple occurrences of formulas, we will need to show the invertibility of more general rules. To show that $(L \rightarrow)$ is $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-invertible, we prove that the following rule is admissible in $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$

$$\frac{[\Gamma_i, [\varphi \rightarrow \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \mid H}{[\Gamma_i, [\psi]^{\lambda_i} \Rightarrow [\varphi]^{\lambda_i}, \Delta_i]_{i=1}^n \mid H}$$

proceeding by induction on the height of a $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivation of $[\Gamma_i, [\varphi \to \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \mid H$. If $\lambda_1 = \cdots = \lambda_n = 0$, then the result follows immediately, so let us assume without loss of generality that $\lambda_1 \geq 1$. Then for the base case, $\Delta_j = \Gamma_j \uplus [\varphi \to \psi]^{\lambda_j}$ for $j \in \{1, \ldots, n\}$, and it suffices to observe that $\vdash_{\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}} \Gamma_j, [\psi]^{\lambda_j} \Rightarrow [\varphi]^{\lambda_j}, \Gamma_j, [\varphi \to \psi]^{\lambda_j} \mid \mathcal{H}$. For the inductive step, we observe that when the last rule applied is not $(\Box_{k,n})$, the claim follows immediately by applying the induction hypothesis, where necessary twice, and the relevant rule. Suppose now that the last rule applied is $(\Box_{k,n})$, so $[\varphi \to \psi]^{\lambda_j}$ must occur also on the right of the sequent as follows:

$$\frac{\Gamma'_0 \Rightarrow \ \mid \mathcal{H} \quad \Gamma'_1 \Rightarrow k[\chi_1] \mid \mathcal{H} \quad \cdots \quad \Gamma'_n \Rightarrow k[\chi_n] \mid \mathcal{H}}{\Omega_j, [\varphi \to \psi]^{\lambda_j}, \Box\Gamma' \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, [\varphi \to \psi]^{\lambda_j}, \Omega_j \mid \mathcal{H}} \ (\Box_{k,n})$$

where $\Gamma_j = \Omega_j \uplus [\varphi \to \psi]^{\lambda_j} \uplus \Box\Gamma'$ and $\Delta_j = \Box\chi_1 \uplus \ldots \uplus \Box\chi_n \uplus [\varphi \to \psi]^{\lambda_j} \uplus \Omega_j$, and also $k\Gamma' = \Gamma'_0 \uplus \Gamma'_1 \uplus \ldots \uplus \Gamma'_n$. Then the claim follows by first applying the induction hypothesis and then applying the rule $(\Box_{k,n})$ and $(R \to)$ (λ_j times) as follows: where $\mathcal{G}$ is obtained from $\mathcal{H}$ by applying induction hypothesis.

$$\dfrac{\dfrac{\Gamma'_0 \Rightarrow \ \mid \mathcal{G} \quad \Gamma'_1 \Rightarrow k\chi_1 \mid \mathcal{G} \quad \cdots \quad \Gamma'_n \Rightarrow k\chi_n \mid \mathcal{G}}{\Omega_j, [\varphi]^{\lambda_j}, [\psi]^{\lambda_j}, \Box\Gamma' \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, [\varphi]^{\lambda_j}, [\psi]^{\lambda_j}, \Omega_j \mid \mathcal{G}} \ (\Box_{k,n})}{\Omega_j, [\psi]^{\lambda_j}, \Box\Gamma' \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, [\varphi]^{\lambda_j}, [\varphi \to \psi]^{\lambda_j}, \Omega_j \mid \mathcal{G}} \ (R \to)(\lambda_j \text{ times})$$

The proof of $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-invertibility of the rule $(R \to)$ is very similar. To show that $(L\wedge)$ is $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-invertible, we prove, more generally, that the following rule is admissible in $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$

$$\frac{[\Gamma_i, [\varphi \wedge \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \mid H}{[\Gamma_i, [\varphi]^{\lambda_i} \Rightarrow \Delta_i \mid \Gamma_i, [\psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \mid H}$$

proceeding by induction on the height of a $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivation of $[\Gamma_i, [\varphi \wedge \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \mid H$. If $\lambda_1 = \cdots = \lambda_n = 0$, then the result follows immediately using (EC), so let us assume without loss of generality that $\lambda_1 \geq 1$. For the base case, $\Delta_j = \Gamma_j \uplus [\varphi \wedge \psi]^{\lambda_j}$ for $j \in \{1, \ldots, n\}$ and it suffices to observe that $\Gamma_j, [\varphi]^{\lambda_j} \Rightarrow \Gamma_j, [\varphi \wedge \psi]^{\lambda_j} \mid \Gamma_j, [\psi]^{\lambda_j} \Rightarrow \Gamma_j, [\varphi \wedge \psi]^{\lambda_j} \mid \mathcal{G}$ is derivable. For example, suppose $\lambda_j = 1$ for $j = 1$, first we have the following derivation:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \Gamma, \varphi, \psi \Rightarrow \Gamma, \Gamma, \psi, \varphi \mid \mathcal{G}}\ (\mathrm{AX})}{\Gamma, \varphi \Rightarrow \Gamma, \psi \mid \Gamma, \psi \Rightarrow \Gamma, \varphi \mid \mathcal{G}}\ (\mathrm{Split}) \qquad \dfrac{}{\Gamma, \varphi \Rightarrow \Gamma, \psi \mid \Gamma, \psi \Rightarrow \Gamma, \psi \mid \mathcal{G}}\ (\mathrm{AX})}{\Gamma, \varphi \Rightarrow \Gamma, \psi \mid \Gamma, \psi \Rightarrow \Gamma, \varphi \wedge \psi \mid \mathcal{G}.}\ (R\wedge)$$

Then, the conclusion is derived as follows:

$$\dfrac{\dfrac{}{\Gamma, \varphi \Rightarrow \Gamma, \varphi \mid \Gamma, \psi \Rightarrow \Gamma, \varphi \wedge \psi \mid \mathcal{G}}\ (\mathrm{AX}) \qquad \Gamma, \varphi \Rightarrow \Gamma, \psi \mid \Gamma, \psi \Rightarrow \Gamma, \varphi \wedge \psi \mid \mathcal{G}}{\Gamma, \varphi \Rightarrow \Gamma, \varphi \wedge \psi \mid \Gamma, \psi \Rightarrow \Gamma, \varphi \wedge \psi \mid \mathcal{G}}\ (R\wedge)$$

For the inductive step, we observe that when the last rule applied is not $(\Box_{k,n})$, the claim follows immediately by applying the induction hypothesis, where necessary twice, and the relevant rule (see e.g. [18] Lemma 5.18 for more details). Suppose now that the last rule applied is $(\Box_{k,n})$, so $[\varphi \wedge \psi]^{\lambda_j}$ must occur also on the right of the sequent as follows:

$$\dfrac{\Gamma'_0 \Rightarrow \ \mid \mathcal{H} \quad \Gamma'_1 \Rightarrow k\chi_1 \mid \mathcal{H} \quad \cdots \quad \Gamma'_n \Rightarrow k\chi_n \mid \mathcal{H}}{\Omega, \Box\Gamma', [\varphi \wedge \psi]^{\lambda_j} \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, [\varphi \wedge \psi]^{\lambda_j} \mid \mathcal{H}}\ (\Box_{k,n})$$

where $\Gamma_j = \Omega \uplus \Box\Gamma'$, and $\Delta_j = \Omega \uplus \Box\chi_1 \uplus \cdots \uplus \Box\chi_n \uplus [\varphi \wedge \psi]^{\lambda_j}$ and also $k\Gamma' = \Gamma'_0 \uplus \ldots \uplus \Gamma'_n$. Then the conclusion is obtained by first applying the induction hypothesis to the premises and then applying $(\Box_{k,n})$, (EW), (Split) and $(R\wedge)$ as required. For example suppose that $\lambda_j = 1$, the claim is derived as follows:

$$\begin{array}{c}\mathcal{D}_1 \\ \dfrac{\dfrac{\Gamma'_0 \Rightarrow \ \mid \mathcal{G} \quad \Gamma'_1 \Rightarrow k\chi_1 \mid \mathcal{G} \quad \cdots \quad \Gamma'_n \Rightarrow k\chi_n \mid \mathcal{G}}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \mid \mathcal{G}}\ (\Box_{k,n})}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \mid \Omega, \Box\Gamma', \psi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \mid \mathcal{G}}\ (\mathrm{EW})\end{array}$$

where $\mathcal{G}$ is obtained from $\mathcal{H} = [\Gamma_i, [\varphi \wedge \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=2}^n \mid H$ by applying induction hypothesis. Similarly, we have

$$\begin{array}{c}\mathcal{D}_2 \\ \Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \mid \Omega, \Box\Gamma', \psi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \psi \mid \mathcal{G}.\end{array}$$

Then, by applying $(R\wedge)$ we have:

$$\dfrac{\mathcal{D}_1 \qquad \mathcal{D}_2}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \mid \Omega, \Box\Gamma', \psi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \wedge \psi \mid \mathcal{G}}\ (R\wedge).$$

Now, by a similar argument, we have:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'_1\\ 2\Gamma'_0 \Rightarrow \;|\, \mathcal{G} \quad 2\Gamma'_1 \Rightarrow 2k\chi_1 \,|\, \mathcal{G} \quad \cdots \quad 2\Gamma'_n \Rightarrow 2k\chi_n \,|\, \mathcal{G}\end{array}}{2\Omega, 2\Box\Gamma', \varphi, \psi \Rightarrow 2\Box\chi_1, \ldots, 2\Box\chi_n, 2\Omega, \varphi, \psi \,|\, \mathcal{G}}\,(\Box_{2k,2n})}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \psi \,|\, \Omega, \Box\Gamma', \psi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \,|\, \mathcal{G}}\,(\text{Split}).$$

And, similar to the derivations $\mathcal{D}_1$ and $\mathcal{D}_2$, we have:

$$\begin{array}{c}\mathcal{D}'_2\\ \Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \psi \,|\, \Omega, \Box\Gamma', \psi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \psi \,|\, \mathcal{G}.\end{array}$$

Now, by applying $(R\wedge)$ we have:

$$\dfrac{\mathcal{D}'_1 \qquad \mathcal{D}'_2}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \psi \,|\, \Omega, \Box\Gamma', \psi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \wedge \psi \,|\, \mathcal{G}.}\,(R\wedge).$$

Finally, again by applying $(R\wedge)$ the claim is obtained. The proof of $\mathbf{HLK(A)}^{\mathrm{r}}$-invertibility of the rule $(R\vee)$ is very similar. To show that $(L\vee)$ is $\mathbf{HLK(A)}^{\mathrm{r}}$-invertible, we prove that the following rules are admissible in $\mathbf{HLK(A)}^{\mathrm{r}}$

$$\dfrac{[\Gamma_i, [\varphi \vee \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \,|\, H}{[\Gamma_i, [\varphi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \,|\, H} \qquad\qquad \dfrac{[\Gamma_i, [\varphi \vee \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \,|\, H}{[\Gamma_i, [\psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=1}^n \,|\, H}$$

proceeding by induction on the height of the derivations of the premises. We only consider the case that the last rule applied in the derivation of the premise is $(\Box_{k,n})$; the other cases are treated easily. Suppose that the last rule applied is $(\Box_{k,n})$, so $[\varphi \vee \psi]^{\lambda_j}$ must occur also on the right of the sequent as follows:

$$\dfrac{\Gamma'_0 \Rightarrow \;|\, \mathcal{H} \quad \Gamma'_1 \Rightarrow k\chi_1 \,|\, \mathcal{H} \quad \cdots \quad \Gamma'_n \Rightarrow k\chi_n \,|\, \mathcal{H}}{\Omega, \Box\Gamma', [\varphi \vee \psi]^{\lambda_j} \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, [\varphi \vee \psi]^{\lambda_j} \,|\, \mathcal{H}}\,(\Box_{k,n}),$$

where $\Gamma_j = \Omega \uplus \Box\Gamma'$, and $\Delta_j = \Omega \uplus \Box\chi_1 \uplus \cdots \uplus \Box\chi_n \uplus [\varphi \wedge \psi]^{\lambda_j}$ and also $k\Gamma' = \Gamma'_0 \uplus \ldots \uplus \Gamma'_n$. Then, for $\lambda_j = 1$, the conclusion is obtained as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma'_0 \Rightarrow \;|\, \mathcal{G} \quad \Gamma'_1 \Rightarrow k\chi_1 \,|\, \mathcal{G} \quad \cdots \quad \Gamma'_n \Rightarrow k\chi_n \,|\, \mathcal{G}}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \,|\, \mathcal{G}}\,(\Box_{k,n})}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \,|\, \Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \psi \,|\, \mathcal{G}}\,(\text{EW})}{\Omega, \Box\Gamma', \varphi \Rightarrow \Box\chi_1, \ldots, \Box\chi_n, \Omega, \varphi \vee \psi \,|\, \mathcal{G}}\,R\vee$$

where $\mathcal{G}$ is obtained from $\mathcal{H} = [\Gamma_i, [\varphi \vee \psi]^{\lambda_i} \Rightarrow \Delta_i]_{i=2}^n \,|\, H$ by applying induction hypothesis. The $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-invertibility of the rule $(R\wedge)$ is proved similarly. $\square$

Lemma 4.4. *The rule* (Mix) *is* $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$*-admissible.*

Proof: To show the $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-admissibility of (Mix), we prove, more generally, that the following rule is admissible

$$\frac{[\Gamma_i \Rightarrow \Delta_i]_{i=1}^n \,|\, H \qquad [\Pi_j \Rightarrow \Sigma_j]_{j=1}^m \,|\, H}{[r_{i_1}\Gamma_i, s_{i_1}\Pi_1 \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=1}^n \,|\cdots|\, [r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=1}^n \,|\, H}$$

for all $r_{i_j}, s_{i_j} \in \mathbb{N} \cup \{0\}$. Proceeding by induction on the lexicographically ordered pair consisting of the sum of the modal depth of the formulas in the premises and the sum of the height of $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $[\Gamma_i \Rightarrow \Delta_i]_{i=1}^n \,|\, H$ and $[\Pi_j \Rightarrow \Sigma_j]_{j=1}^m \,|\, H$, respectively. If $\mathcal{D}_1$ and $\mathcal{D}_2$ have height 0, then $[\Gamma_i \Rightarrow \Delta_i]_{i=1}^n \,|\, H$ and $[\Pi_j \Rightarrow \Sigma_j]_{j=1}^m \,|\, H$ are instances of (AX). i.e., $\Gamma_i = \Delta_i$ for some $1 \leqslant i \leqslant n$, and $\Pi_j = \Sigma_j$ for some $1 \leqslant j \leqslant m$, (in particular if Γ_i, Δ_i, Π_j, and Σ_j contain only variables), then $r_{i_j}\Gamma_i \uplus s_{i_j}\Pi_j = r_{i_j}\Delta_i \uplus s_{i_j}\Sigma_j$ and so $[r_{i_1}\Gamma_i, s_{i_1}\Pi_1 \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=1}^n \,|\cdots|\, [r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=1}^n \,|\, H$ is an instance of (AX). If the last application of rules in $\mathcal{D}_1$ and $\mathcal{D}_2$ are not $(\Box_{k,n})$ then the result follows easily by one (or two) applications of the induction hypothesis and further applications of the rule. For example, suppose $\mathcal{D}_2$ ends with

$$\frac{\Pi', \varphi \Rightarrow \Sigma_1 \,|\, \Pi', \psi \Rightarrow \Sigma_1 \,|[\Pi_j \Rightarrow \Sigma_j]_{j=2}^m|\, H}{\Pi', \varphi \wedge \psi \Rightarrow \Sigma_1 |[\Pi_j \Rightarrow \Sigma_j]_{j=2}^m \,|\, H}\ (L\wedge),$$

where $\Pi_1 = \Pi' \uplus [\varphi \wedge \psi]$. An application of the induction hypothesis to the $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivation of the premise $[\Gamma_i \Rightarrow \Delta_i]_{i=1}^n$ together with a $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivation of

$$\Pi', \varphi \Rightarrow \Sigma_1 \,|\, \Pi', \psi \Rightarrow \Sigma_1 \,|[\Pi_j \Rightarrow \Sigma_j]_{j=2}^m|\, H$$

yields

$$\begin{aligned}&[r_{i_1}\Gamma_i, s_{i_1}\Pi', s_{i_1}\varphi \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=1}^n \,|\, [r_{i_1}\Gamma_i, s_{i_1}\Pi', s_{i_1}\psi \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=1}^n| \\ &\qquad \cdots |[r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=1}^n \,|\, H.\end{aligned}$$

It follows then that the following hypersequent is $\mathbf{HLK(A)}^{\mathrm{r}}$-derivable using $\sum_{i=1}^{n} s_{i_1}$ times applications of the rule $(L\wedge)$:

$$[r_{i_1}\Gamma_i, s_{i_1}\Pi', s_{i_1}(\varphi\wedge\psi) \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=1}^{n} \,|\cdots|[r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=1}^{n} \,|\, H.$$

The case where $\mathcal{D}_2$ ends with $(R\vee)$, $(L\to)$, $(R\to)$, (EC) or (Split) is treated by a similar argument. If $\mathcal{D}_2$ ends with

$$\frac{\Pi', \varphi \Rightarrow \Sigma_1 \,|[\Pi_j \Rightarrow \Sigma_j]_{j=2}^{m}|\, H \qquad \Pi', \psi \Rightarrow \Sigma_1 \,|[\Pi_j \Rightarrow \Sigma_j]_{j=2}^{m}|\, H}{\Pi', \varphi\vee\psi \Rightarrow \Sigma_1|[\Pi_j \Rightarrow \Sigma_j]_{j=2}^{m} \,|\, H}\,(L\vee),$$

where $\Pi_1 = \Pi' \uplus [\varphi\vee\psi]$. Then, by the induction hypothesis,

$$\vdash_{\mathbf{HLK(A)}}[r_{i_1}\Gamma_i, s_{i_1}\Pi', s_{i_1}\varphi \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=1}^{n}|\cdots |[r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=1}^{n} \,|\, H$$

$$\vdash_{\mathbf{HLK(A)}}\, [r_{i_1}\Gamma_i, s_{i_1}\Pi', s_{i_1}\psi \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=1}^{n}|\cdots |[r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=1}^{n} \,|\, H$$

So, the conclusion is derived by $\sum_{i=1}^{n} s_{i_1}$ times applications of $(L\vee)$. The case where $\mathcal{D}_2$ ends with $(R\wedge)$ is treated by a similar argument. Finally, let us consider the case where $\mathcal{D}_1$ ends with an application of $(\Box_{k,p})$ as follows:

$$\frac{\Gamma_0 \Rightarrow \,|\,\mathcal{G} \quad \Gamma_1 \Rightarrow k\varphi_1 \,|\, \mathcal{H} \cdots \Gamma_p \Rightarrow k\varphi_p \,|\, \mathcal{H}}{\Omega, \Box\Gamma' \Rightarrow \Box\varphi_1, \ldots, \Box\varphi_p, \Omega \,|\, \mathcal{H}}\,(\Box_{k,p}),$$

where $\Gamma_1 = \Omega \uplus \Box\Gamma'$ and $\Delta_1 = [\Box\varphi_1] \uplus \ldots \uplus [\Box\varphi_p] \uplus \Omega$, in addition $k\Gamma' = \Gamma_0 \uplus \ldots \uplus \Gamma_p$ and $\mathcal{H} = [\Gamma_i \Rightarrow \Delta_i]_{i=2}^{n} \,|\, H$, and suppose $\mathcal{D}_2$ ends with

$$\frac{\Pi_0 \Rightarrow |\mathcal{H} \quad \Pi_1 \Rightarrow l\psi_1 \,|\mathcal{H} \cdots \Pi_q \Rightarrow l\psi_q|\mathcal{H}}{\Theta, \Box\Pi' \Rightarrow \Box\psi_1, \ldots, \Box\psi_q, \Theta|\mathcal{H}}\,(\Box_{l,q}),$$

where $\Pi_1 = \Theta\uplus\Box\Pi'$ and $\Sigma_1 = [\Box\psi_1]\uplus\ldots\uplus[\Box\psi_p]\uplus\Theta$, in addition $l\Pi' = \Pi_0\uplus\ldots\uplus\Pi_q$ and $\mathcal{H} = [\Pi_j \Rightarrow \Sigma_j]_{j=2}^{m} \,|\, H$. Then, applying the rule $(\Box_{kl, r_{1_1}p+s_{1_1}q})$, we obtain the required $\mathbf{HLK(A)}^{\mathrm{r}}$-derivation

$$\frac{r_{1_1}l\Gamma_0, s_{1_1}k\Pi_0 \Rightarrow \,|\,\mathcal{G} \qquad \{l\Gamma_i \Rightarrow kl\varphi_i \,|\, \mathcal{G}\}_{i=1}^{r_{1_1}p} \cdots \{k\Pi_j \Rightarrow kl\psi_j \,|\, \mathcal{G}\}_{j=1}^{s_{1_1}q}}{\begin{array}{l} r_{1_1}\Omega, s_{1_1}\Theta, r_{1_1}\Box\Gamma', s_{1_1}\Box\Pi' \Rightarrow \\ r_{1_1}\Box\varphi_1, \ldots, r_{1_1}\Box\varphi_p, s_{1_1}\Box\psi_1, \ldots, s_{1_1}\Box\psi_q, r_{1_1}\Omega, s_{1_1}\Theta \,|\, \mathcal{G}\end{array}}$$

where,

$$\mathcal{G} = [r_{i_1}\Gamma_i, s_{i_1}\Pi_1 \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=2}^{n} | \cdots \\ | [r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=2}^{n} \,|\, H$$

and the premises are all $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivable using the induction hypothesis. For example,

$$r_{1_1} l\Gamma_0, s_{1_1} k\Pi_0 \Rightarrow \; | [r_{i_1}\Gamma_i, s_{i_1}\Pi_1 \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=2}^{n} | \cdots \\ | [r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=2}^{n} | \, H$$

is derived as follows using the induction hypothesis (note that $r_{i_j}, s_{i_j} \in \mathbb{N} \cup \{0\}$):

$$\frac{\Gamma_0 \Rightarrow \; | \, [\Gamma_i \Rightarrow \Delta_i]_{i=2}^{n} \, | \, H \qquad \Pi_0 \Rightarrow \; | \, [\Pi_j \Rightarrow \Sigma_j]_{j=2}^{m} \, | \, H}{\begin{array}{l} r_{1_1} l\Gamma_0, s_{1_1} k\Pi_0 \Rightarrow \; | [r_{i_1}\Gamma_i, s_{i_1}\Pi_1 \Rightarrow r_{i_1}\Delta_i, s_{i_1}\Sigma_1]_{i=2}^{n} | \cdots \\ | [r_{i_m}\Gamma_i, s_{i_m}\Pi_m \Rightarrow r_{i_m}\Delta_i, s_{i_m}\Sigma_m]_{i=2}^{n} | \, H \end{array}}$$

□

Theorem 4.5. **HLK(A)** *admits cut-elimination.*

Proof: To establish cut-elimination for **HLK**(**A**), it suffices to prove that an uppermost application of (Cut) in a **HLK**(**A**)-derivation can be eliminated; that is, we show that cutfree **HLK**(**A**)-derivations of the premises of an instance of (Cut) can be transformed into a cut-free **HLK**(**A**)-derivation of the conclusion. Observe first that the rule $(\Box_n)$ is $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivable using $(\Box_{k,n})$ with $k = n, \varphi_1 = \cdots = \varphi_n = \varphi$ and $\Gamma_1 = \ldots = \Gamma_n = \Gamma$. Hence, the proof of Lemma 4.4 shows that any cut-free **HLK**(**A**)-derivation can be transformed algorithmically into a $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivation. We prove (constructively) that the following rule called "cancellation" rule is $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-admissible:

$$\frac{[\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=1}^{n} \,|\, H}{[\Gamma_i \Rightarrow \Delta_i]_{i=1}^{n} \,|\, H} \text{ (CAN)}.$$

Suppose then that there are cut-free **HLK**(**A**)-derivations of the premises $\Gamma, \varphi \Rightarrow \Delta \,|\, H$ and $\Pi \Rightarrow \varphi, \Sigma \,|\, H$ of an uppermost application of (Cut). By (Mix), we obtain a cut-free **HLK**(**A**)- derivation of $\Gamma, \Pi, \varphi \Rightarrow \varphi, \Delta, \Sigma \,|\, H$ and hence a $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivation of this sequent. By cancellation rule, we obtain a $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivation of $\Gamma, \Pi \Rightarrow \Delta, \Sigma \,|\, H$, which also gives the desired cut-free **HLK**(**A**)-derivation. We prove the admissibility of the cancellation rule by induction on the lexicographically ordered triple consisting of the sum of the modal depth of the formulas φ_i, $1 \leq i \leq n$, sum of the complexities of the formulas φ_i, $1 \leq i \leq n$, and the height of the derivation of the premise. For the base case, suppose that the formulas φ_i for all $1 \leq i \leq n$ are variables. If the premise is an instance of (AX), then $\Gamma_i, \varphi_i = \varphi_i, \Delta_i$ for some $1 \leq i \leq n$, i.e., $\Gamma_i = \Delta_i$ and so $[\Gamma_i \Rightarrow \Delta_i]_{i=1}^n$ is an instance of (AX). We observe that when the last rule applied is not $(\Box_{k,n})$, the claim follows immediately by applying the induction hypothesis and, where necessary, the relevant rule. Let us consider some cases; suppose that the last rule applied is $(L \to)$ as follows:

$$\frac{\Gamma_1', \varphi_1, \chi \Rightarrow \psi, \varphi_1, \Delta_1 \mid [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \mid H}{\psi \to \chi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \mid H} \ (L \to),$$

where $\Gamma_1 = \psi \to \chi, \Gamma_1'$. Then, the height of the premise is reduced and so by applying the induction hypothesis the conclusion is obtained as follows:

$$\frac{\dfrac{\Gamma_1', \varphi_1, \chi \Rightarrow \psi, \varphi_1, \Delta_1 \mid [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \mid H}{\Gamma_1', \chi \Rightarrow \psi, \Delta_1 \mid [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid H} \ (\mathrm{IH})}{\psi \to \chi, \Gamma_1' \Rightarrow \Delta_1 \mid [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid H} \ (L \to).$$

The cases where the last rule applied is $(R \to)$ or (Split) are very similar. Suppose that the last rule applied is $(L\wedge)$ as follows:

$$\frac{\psi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid \chi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \mid H}{\psi \wedge \chi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \mid H} \ (L\wedge),$$

where $\Gamma_1 = \psi \wedge \chi, \Gamma_1'$. Then, the height of the premise is reduced and so by applying the induction hypothesis we have:

$$\frac{\psi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid \chi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \mid H}{\psi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid \chi, \Gamma_1', \Rightarrow \Delta_1 \mid [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid H} \ (\mathrm{IH})$$

Therefore, the sum of the complexities of the formulas φ_i is reduced, again by applying the induction hypothesis the conclusion is obtained as follows:

$$\frac{\dfrac{\psi, \Gamma_1', \varphi_1 \Rightarrow \varphi_1, \Delta_1 \mid \chi, \Gamma_1', \Rightarrow \Delta_1 \mid [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid H}{\psi, \Gamma_1' \Rightarrow \Delta_1 \mid \chi, \Gamma_1', \Rightarrow \Delta_1 \mid [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid H} \ (\mathrm{IH})}{\psi \wedge \chi, \Gamma_1' \Rightarrow \Delta_1 \mid [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \mid H} \ (L\wedge).$$

The cases where the last rule applied is $(R\vee)$ or (EC) are very similar. Suppose now that the last rules applied is $(\Box_{k,m})$ as follows:

$$\frac{\Pi_0 \Rightarrow \;|\, \mathcal{H} \quad \Pi_1 \Rightarrow k\psi_1 \,|\, \mathcal{H} \quad \cdots \quad \Pi_m \Rightarrow k\psi_m \,|\, \mathcal{H}}{\Sigma, \Box\Pi, \varphi_1 \Rightarrow \varphi_1, \Box\psi_1, \ldots, \Box\psi_m, \Sigma \,|\, \mathcal{H}}\,(\Box_{k,m}),$$

where $\mathcal{H} = [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \,|\, H$, and $k\Pi = \Pi_0 \uplus \Pi_1 \uplus \ldots \uplus \Pi_m$, in addition $k = k_0 + k_1 + \ldots + k_m$. Thus, the sum of the complexities of the formulas φ_i is reduced, by applying the induction hypothesis we have $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivations of

$$\begin{array}{l} \Pi_0 \Rightarrow \;|\, [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H \\ \Pi_1 \Rightarrow k\psi_1 \,|\, [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H \\ \vdots \\ \Pi_m \Rightarrow k\psi_m \,|\, [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H \end{array}$$

Then, by applying the rule $(\Box_{k,m})$, we have a $\mathbf{HLK}(\mathbf{A})^{\mathrm{r}}$-derivation of

$$\Sigma, \Box\Pi, \Rightarrow \Box\psi_1, \ldots, \Box\psi_m, \Sigma \,|\, [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H$$

For the inductive step, suppose that $\varphi_i = \psi \to \chi$ for some $1 \le i \le n$, then we use the invertibility of $(L \to)$ and $(R \to)$ and apply the induction hypothesis twice. If φ_i has the form $\psi \wedge \chi$ for some $1 \le i \le n$, then the conclusion is obtained as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_1, \psi \wedge \chi \Rightarrow \psi \wedge \chi, \Delta_1 \,|\, [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \,|\, H}{\Gamma_1, \psi \Rightarrow \psi \wedge \chi, \Delta_1 \,|\, \Gamma_1, \chi \Rightarrow \psi \wedge \chi, \Delta_1 \,|\, [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \,|\, H}\,(L\wedge^{-1})}{\Gamma_1, \psi \Rightarrow \psi, \Delta_1 \,|\, \Gamma_1, \chi \Rightarrow \chi, \Delta_1 \,|\, [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \,|\, H}\,(R\wedge^{-1}) \text{ twice}}{\Gamma_1 \Rightarrow \Delta_1 \,|\, \Gamma_1 \Rightarrow \Delta_1 \,|\, [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H}\,(\mathrm{IH}) \text{ twice}}{\Gamma_1 \Rightarrow \Delta_1 \,|\, [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H}\,(\mathrm{EC})$$

Note that by applying the invertibility of the logical rules the height and sum of the complexities of the formulas in the premise can increase, but the sum of the complexities of the formulas φ_i is reduced. The cases where φ_i for some $1 \le i \le n$ has the form $\psi \vee \chi$ are very similar. Lastly, suppose that $\varphi_i = \Box\chi$ for some $1 \le i \le n$, and the derivation ends with

$$\frac{\Pi_0, k_0\chi \Rightarrow \;|\, \mathcal{H} \quad \Pi_1, k_1\chi \Rightarrow k\chi \,|\, \mathcal{H} \quad \{\Pi_i, k_i\chi \Rightarrow k\psi_i \,|\, \mathcal{H}\}_{i=2}^p}{\Sigma, \Box\Pi, \Box\chi \Rightarrow \Box\chi, \Box\psi_2, \ldots, \Box\psi_n, \Sigma \,|\, \mathcal{H}}\,(\Box_{k,p}),$$

where $\mathcal{H} = [\Gamma_i, \varphi_i \Rightarrow \varphi_i, \Delta_i]_{i=2}^n \,|\, H$, and $k\Pi = \Pi_0 \uplus \Pi_1 \uplus \ldots \uplus \Pi_p$, in addition $k = k_0 + k_1 + \ldots + k_p$. In this case, the sum of the modal depth of the formulas φ_i is reduced. By the induction hypothesis, $\vdash_{\mathbf{HLK(A)}^{\mathrm{r}}} \Pi_1 \Rightarrow (k - k_1)\chi \,|\, \mathcal{H}$. By the $\mathbf{HLK(A)}^{\mathrm{r}}$-admissibility of the rule (mix), we have $\mathbf{HLK(A)}^{\mathrm{r}}$-derivations of

$$k_0\Pi_1, (k - k_1)\Pi_0, (k - k_1)k_0\chi \Rightarrow (k - k_1)k_0\chi \,|\, \mathcal{H}$$

$$k_i\Pi_1, (k-k_1)\Pi_i, (k-k_1)k_i\chi \Rightarrow (k-k_1)k_i\chi, (k-k_1)k\psi_i \,|\, \mathcal{H} \text{ for } i \in \{2, \ldots, p\}.$$

So, by the induction hypothesis, we have $\mathbf{HLK(A)}^{\mathrm{r}}$-derivations of

$$k_0\Pi_1, (k - k_1)\Pi_0 \Rightarrow \,|\, \mathcal{G}$$

$$k_i\Pi_1, (k - k_1)\Pi_i \Rightarrow (k - k_1)k\psi_i \,|\, \mathcal{G} \text{ for } i \in \{2, \ldots, p\},$$

where $\mathcal{G} = [\Gamma_i \Rightarrow \Delta_i]_{i=2}^n \,|\, H$. Now by an application of $(\Box_{((k-k_1)k,n-1)})$, we have a $\mathbf{HLK(A)}^{\mathrm{r}}$-derivation ending with

$$\frac{k_0\Pi_1, (k - k_1)\Pi_0 \Rightarrow \,|\, \mathcal{G} \qquad \{k_i\Pi_1, (k - k_1)\Pi_i \Rightarrow (k - k_1)k\psi_i \,|\, \mathcal{G}\}_{i=2}^p}{\Sigma, \Box\Pi \Rightarrow \Box\psi_2, \ldots, \Box\psi_n, \Sigma \,|\, \mathcal{G}}$$

where $(k - k_1)k\Pi = (k_0 + k_2 + \cdots + k_p)(\Pi_0 \uplus \Pi_1 \uplus \ldots \uplus \Pi_p)$. □

We now turn our attention to showing that the axiomatic and hypersequent presentations really characterize the same logics, writing $+\{\varphi_1, \ldots, \varphi_n\}$ as shorthand for $\varphi_1 + \ldots + \varphi_n$.

LEMMA 4.6.

(i) *If* $\vdash_{\mathbf{HLK(A)}} \Gamma, \varphi + \psi \Rightarrow \Delta \,|\, H$, *then* $\vdash_{\mathbf{HLK(A)}} \Gamma, \varphi, \psi \Rightarrow \Delta \,|\, H$.

(ii) *If* $\vdash_{\mathbf{HLK(A)}} \Gamma \Rightarrow \Delta, \varphi + \psi \,|\, H$, *then* $\vdash_{\mathbf{HLK(A)}} \Gamma \Rightarrow \Delta, \varphi, \psi \,|\, H$.

PROOF: For (i), since $\vdash_{\mathbf{HLK(A)}} \varphi, \psi \Rightarrow \varphi + \psi \,|\, H$, if $\vdash_{\mathbf{HLK(A)}} \Gamma, \varphi + \psi \Rightarrow \Delta \,|\, H$, then by (Cut), $\vdash_{\mathbf{HLK(A)}} \Gamma, \varphi, \psi \Rightarrow \Delta \,|\, H$. The case (ii) is similar. □

LEMMA 4.7. *If* $\vdash_{\mathbf{HLK(A)}} \Rightarrow \mathcal{I}(H)$, *then* $\vdash_{\mathbf{HLK(A)}} H$.

PROOF: Let $H = \Gamma_1 \Rightarrow \Delta_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Delta_n$. If

$$\vdash_{\mathbf{HLK(A)}} (\sum \Gamma_1 \to \sum \Delta_1) \vee \cdots \vee (\sum \Gamma_n \to \sum \Delta_n),$$

then by invertibility of the rules $(R\vee)$ and $(R\to)$,

$$\vdash_{\mathbf{HLK(A)}} (\sum \Gamma_1 \Rightarrow \sum \Delta_1) \,|\, \cdots \,|\, (\sum \Gamma_n \Rightarrow \sum \Delta_n).$$

Hence, by Lemma 4.6, $\vdash_{\mathbf{HLK(A)}} H$. □

THEOREM 4.8. *$\vdash_{\mathbf{HLK(A)}} H$ iff $\vdash_{\mathbf{LK(A)}} \mathcal{I}(H)$.*

PROOF: For the left-to-right direction we proceed by induction on the height of the derivation of H in $\mathbf{HLK}(\mathbf{A})$. If H is an instance of an axiom of $\mathbf{HLK}(\mathbf{A})$, then it is easy to check that $\vdash_{\mathbf{LK(A)}} \mathcal{I}(H)$. For the inductive step, suppose that H follows by some rule of $\mathbf{HLK}(\mathbf{A})$ from $H_1, \ldots, H_n$. By the induction hypothesis n times, we have $\vdash_{\mathbf{LK(A)}} \mathcal{I}(H_1), \ldots, \vdash_{\mathbf{LK(A)}} \mathcal{I}(H_n)$. For the non-modal rules of $\mathbf{HLK}(\mathbf{A})$ (see e.g. [18] for details), it is easy to check that

$$\vdash_{\mathbf{LK(A)}} \mathcal{I}(H_1) \to (\mathcal{I}(H_2) \to (\cdots \to (\mathcal{I}(H_n) \to \mathcal{I}(H))) \cdots)$$

and that hence, by (mp) n times, $\vdash_{\mathbf{LK(A)}} \mathcal{I}(H)$. For the modal rule, suppose that $\vdash_{\mathbf{LK(A)}} \mathcal{I}(H) \vee \mathcal{I}(\Gamma \Rightarrow n\varphi)$. By Theorem 3.3, it is sufficient to show that $\mathcal{I}(\Box\Gamma \Rightarrow n\Box\varphi \,|\, H)$ is valid in every $\mathbf{LK(A)}$-algebra. Consider a valuation v for such an algebra. Either $v(\mathcal{I}(H)) \geq \bar{0}$ and hence $v(\mathcal{I}(H) \vee \mathcal{I}(\Box\Gamma \Rightarrow n\Box\varphi) \geq \bar{0}$ or $v(\mathcal{I}(\Gamma \Rightarrow n\varphi)) \geq \bar{0}$. If the latter, then $I(v(\mathcal{I}(\Gamma \Rightarrow n\varphi)) \geq I(\bar{0}))$. But $I(v(\mathcal{I}(\Gamma \Rightarrow n\varphi))) = v(\mathcal{I}(\Box\Gamma \Rightarrow n\Box\varphi))$ so we are done.
For the right-to-left direction, we have (an easy exercise) that the axioms of $\mathbf{LK}(\mathbf{A})$ are derivable in $\mathbf{HLK}(\mathbf{A})$. Moreover, (nec) corresponds to $(\Box)$, (adj) corresponds to $(\mathrm{R}\wedge)$, and (mp) can be derived from $\vdash_{\mathbf{HLK(A)}} \Rightarrow \varphi$ and $\vdash_{\mathbf{HLK(A)}} \Rightarrow \varphi \to \psi$, by using (Cut) twice with $\vdash_{\mathbf{HLK(A)}} \varphi, \varphi \to \psi \Rightarrow \psi$. Hence, if $\vdash_{\mathbf{LK(A)}} \mathcal{I}(H)$, then $\vdash_{\mathbf{HLK(A)}} \Rightarrow \mathcal{I}(H)$, and so by Lemma 4.7, $\vdash_{\mathbf{HLK(A)}} \Rightarrow H$. □

5. Concluding remarks

The paper is devoted to a proof-theoretic account of continuous modal logics: many-valued modal logics with connectives interpreted locally by continuous functions over sets of real numbers [10]. I have introduced linear abelian modal logic $\mathbf{LK}(\mathbf{A})$, which is an extension of the abelian modal logic $\mathbf{K}(\mathbf{A})$, where propositional connectives are interpreted using lattice ordered group operations over the real numbers. I have provided a hypersequent calculus admitting cut-elimination for $\mathbf{LK}(\mathbf{A})$. Moreover, the correspondence between this calculus and the complete axiomatization with respect to both appropriate algebras and linearly ordered algebras is established.

I have only focused in this work on the extension of the sequent calculus for the modal multiplicative fragment of $\mathbf{K}(\mathbf{A})$ to a hypersequent calculus for the full logic. Clearly, there are many open questions still to be addressed. The most pressing issue is to provide a suitable Kripke model for $\mathbf{LK}(\mathbf{A})$ and prove the completeness theorem with respect to it. It seems that adapting the Kripke semantics and prove completeness with respect to the Kripke semantics is more tricky. Since the distributivity of box over the operator "+", i.e., $\Box(\varphi + \psi) \to \Box\varphi + \Box\psi$ is not derivable in the provided hypersequent calculus, this formula should not be valid in Kripke models. Therefore, it seems that we need some conditions on the accessibility relation in the Kripke models in which the formula $\Box(\varphi \vee \psi) \to \Box\varphi \vee \Box\psi$ is valid, while the formula $\Box(\varphi + \psi) \to \Box\varphi + \Box\psi$ is not valid.

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Hamzeh Mohammadi
Isfahan University of Technology
Department of Mathematical Sciences
Postal Code: 841583111
Isfahan, Iran
e-mail: hamzeh.mohammadi@math.iut.ac.ir