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# ON HOMOMORPHISM AND CARTESIAN PRODUCTS OF INTUITIONISTIC FUZZY PMS-SUBALGEBRA OF A PMS-ALGEBRA

**Abstract**

In this paper, we introduce the notion of intuitionistic fuzzy PMS-subalgebras under homomorphism and Cartesian product and investigate several properties. We study the homomorphic image and inverse image of the intuitionistic fuzzy PMS-subalgebras of a PMS-algebra, which are also intuitionistic fuzzy PMS-subalgebras of a PMS-algebra, and find some other interesting results. Furthermore, we also prove that the Cartesian product of intuitionistic fuzzy PMS-subalgebras is again an intuitionistic fuzzy PMS-subalgebra and characterize it in terms of its level sets. Finally, we consider the strongest intuitionistic fuzzy PMS-relations on an intuitionistic fuzzy set in a PMS-algebra and demonstrate that an intuitionistic fuzzy PMS-relation on an intuitionistic fuzzy set in a PMS-algebra is an intuitionistic fuzzy PMS-subalgebra if and only if the corresponding intuitionistic fuzzy set in a PMS-algebra is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra.

*Keywords*: PMS-algebra, intuitionistic fuzzy PMS-subalgebra, homomorphism, cartesian product and strongest intuitionistic fuzzy relation.

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## 1. Introduction

In 1965, Zadeh [12] introduced the fundamental concept of a fuzzy set as an extension of the classical set theory for representing uncertainties in a physical world. Following the introduction of a fuzzy set, several researchers undertook a large number of studies on the extension of a fuzzy set. Atanassov [2, 3] investigated an intuitionistic fuzzy set as an extension of a fuzzy set to deal with uncertainties more efficiently in the actual situation. In 2007, Panigrahi and Nanda [5] introduced the idea of an intuitionistic fuzzy relation between any two intuitionistic fuzzy subsets defined in the given universal sets. In 2011, Anitha and Arjunan [1] studied the strongest intuitionistic fuzzy relations on intuitionistic fuzzy ideals of Hemirings and obtained some interesting results. In 2016, Sithar Selvam and Nagalakshmi [8] introduced a new class of algebra called PMS-algebra. Sithar Selvam and Nagalakshmi [7] fuzzified PMS-subalgebras and PMS ideals in PMS-algebra. In the same year, Sithar Selvam and Nagalakshmi [9] also introduced the concept of homomorphism and Cartesian product of fuzzy PMS-algebra and set up some properties. In our earlier paper [4], we introduced the notion of fuzzy PMS-subalgebra in PMS-algebra and studied some of its properties.
In this paper, we discuss the notion of intuitionistic fuzzy PMS-subalgebras under homomorphism and Cartesian product and investigate several properties. Furthermore, we investigate the homomorphic image and the inverse image of the intuitionistic fuzzy PMS-subalgebras of a PMS-algebra and find some results. Finally, we consider the strongest intuitionistic fuzzy PMS-relations on an intuitionistic fuzzy set in a PMS-algebra and demonstrate that an intuitionistic fuzzy PMS-relation on an intuitionistic fuzzy set in a PMS-algebra is an intuitionistic fuzzy PMS-subalgebra if and only if the corresponding intuitionistic fuzzy set in a PMS-algebra is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra.

## 2. Preliminaries

In this section, we recall some basic definitions and results that are used in the study of this paper.

Definition 2.1 ([8]). A nonempty set X with a constant 0 and a binary operation '$*$' is called a PMS-algebra if it satisfies the following axioms.

1. $0 * x = x$
2. $(y * x) * (z * x) = z * y$, for all $x, y, z \in X$.

For $x, y \in X$ , we define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$.

DEFINITION 2.2 ([8]). Let $S$ be a nonempty subset of a PMS-algebra $X$. Then S is called a PMS-subalgebra of $X$ if $x * y \in S$, for all $x, y \in S$.

DEFINITION 2.3 ([7, 9]). Let $X$ and $Y$ be any two PMS- algebras. Then a mapping $f : X \to Y$ is said to be a homomorphism of PMS-algebras if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. $f$ is called an epimorphism if it is onto and endomorphism if $f$ is a mapping from a PMS-algebra $X$ to itself.

Note: If $f$ is a homomorphism of PMS-algebra, then $f(0) = 0$.

DEFINITION 2.4 ([12]). Let $X$ be a nonempty set. A fuzzy set A in $X$ is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ represents the degree of membership of $x$ in $X$.

DEFINITION 2.5 ([7]). A fuzzy set A in a PMS-algebra $X$ is called fuzzy PMS-subalgebra of $X$ if $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$, for all $x, y \in X$.

DEFINITION 2.6 ([2, 3]). An intuitionistic fuzzy subset A in a nonempty set $X$ is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ define the degree of membership and the degree of nonmembership respectively and satisfying the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$.

*Remark* 2.7. Ordinary fuzzy sets over $X$ may be viewed as special intuitionistic fuzzy sets with the nonmembership function $\nu_A(x) = 1 - \mu_A(x)$. So each Ordinary fuzzy set may be written as $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in X\}$ to define an intuitionistic fuzzy set. For the sake of simplicity we write $A = (\mu_A, \nu_A)$ for an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$.

DEFINITION 2.8 ([2, 3]). Let A and B be intuitionistic fuzzy subsets of $X$, where $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in X\}$, then

1. $A \cap B = \{\langle x, min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}\rangle \,| x \in X\}$
2. $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \,| x \in X\}$
3. $\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \,| x \in X\}$

Definition 2.9 ([4]). An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS -algebra $X$ is called an intuitionistic fuzzy PMS-subalgebra of $X$ if $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}, \forall x, y \in X$

Definition 2.10 ([10]). Let $X$ and $Y$ be any two nonempty sets and $f : X \to Y$ be a mapping. If $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are intuitionistic fuzzy subsets of $X$ and $Y$ respectively. Then the image of $A$ under $f$ is defined as $f(A) = \left\{\left\langle y, \mu_{f(A)}(y), \nu_{f(A)}(y)\right\rangle | y \in Y\right\}$, where

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x) & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and

$$\nu_{f(A)}(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \nu_A(x) & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

The inverse image of $B$ under $f$ is denoted by $f^{-1}(B)$ and is defined as

$$f^{-1}(B)(x) = \left\{\left\langle x, \mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x)\right\rangle | x \in X\right\},$$

where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$ for all $x \in X$.

Definition 2.11 ([10]). An intuitionistic fuzzy subset $A$ in a nonempty set $X$ with the degree of membership $\mu_A : X \to [0, 1]$ and the degree of non membership $\nu_A : X \to [0, 1]$ is said to have sup-inf property, if for any subset $T \subseteq X$ there exists $x_0 \in T$ such that $\mu_A(x_0) = \sup\limits_{t \in T} \mu_A(t)$ and $\nu_A(x_0) = \inf\limits_{t \in T} \mu_A(t)$

Definition 2.12. [5, 11] Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two intuitionistic fuzzy subsets of $X$ and $Y$ respectively. Then the Cartesian product of $A$ and $B$ is defined as

$$A \times B = \{\langle (x, y), \mu_{A \times B}(x, y), \nu_{A \times B}(x, y)\rangle | x \in X, y \in Y\},$$

where $\mu_{A \times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$ and $\nu_{A \times B}(x, y) = \max\{(\nu_A(x), \nu_B(x))\}$ such that $\mu_{A \times B} : X \times Y \to [0, 1]$ and $\nu_{A \times B} : X \times Y \to [0, 1]$, for all $x \in X$ and $y \in Y$.

*Remark* 2.13. Let $X$ and $Y$ be PMS-algebras, for all $(x, y), (u, v) \in X \times Y$, we define '$*$' on $X \times Y$ by $(x, y)*(u, v) = (x*u, y*v)$. Clearly $(X \times Y; *, (0, 0))$ is a PMS-algebra.

DEFINITION 2.14. [5] A fuzzy relation $A$ on a nonempty set $X$ is a fuzzy set $A$ with a membership function $\mu_A : X \times X \to [0, 1]$.

DEFINITION 2.15. [6, 5] An intuitionistic fuzzy relation $R$ on a non empty set $X$ is an expression of the form $R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \,|x, y \in X\}$ where $\mu_R : X \times X \to [0, 1]$ and $\nu_R : X \times X \to [0, 1]$ satisfy the condition $0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1$ for every $(x, y) \in X \times X$.

DEFINITION 2.16 ([1, 6, 5]). Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set on a set $X$ and $R = (\mu_R, \nu_R)$ is an intuitionistic fuzzy relation on a set $X$. Then the strongest intuitionistic fuzzy relation $R_A$ on X, that is, an intuitionistic fuzzy relation $R$ on $A$ whose membership function $\mu_{R_A} : X \times X \to [0, 1]$ and whose nonmembership function $\nu_{R_A} : X \times X \to [0, 1]$ are given by $\mu_{R_A}(x, y) = \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_{R_A}(x, y) = \max\{\nu_A(x), \nu_A(y)\}$.

## 3. Homomorphism on intuitionistic Fuzzy PMS-subalgebras

In this section, we discuss on intuitionistic fuzzy PMS-subalgebras in a PMS-algebra under homomorphism. The homomorphic image and inverse image of intuitionistic fuzzy PMS-subalgebras of a PMS-algebra, as well as other results, are examined. Unless otherwise stated, $X$ and $Y$ refer to a PMS-algebra throughout this and the following section.

THEOREM 3.1. *Let $f : X \to Y$ be an epimorphism of PMS-algebras. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$ with sup-inf property, then $f(A)$ is an intuitionistic fuzzy PMS-subalgebra of $Y$.*

PROOF: Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$ and let $a, b \in Y$ with $x_0 \in f^{-1}(a)$ and $y_0 \in f^{-1}(b)$ such that

$$\mu_A(x_0) = \sup_{x \in f^{-1}(a)} \mu_A(x),\ \mu_A(y_0) = \sup_{x \in f^{-1}(b)} \mu_A(x),$$

and

$$\nu_A(x_0) = \inf_{x \in f^{-1}(a)} \nu_A(x),\ \nu_A(y_0) = \inf_{x \in f^{-1}(b)} \nu_A(x),$$

then by Definition 2.10 and 2.11 we have

$$\begin{aligned}\mu_{f(A)}(a*b) = \sup_{x\in f^{-1}(a*b)} \mu_A(x) &= \mu_A(x_0 * y_0)\\ &\geq \min\{\mu_A(x_0), \mu_A(y_0)\}\\ &= min\{\sup_{x\in f^{-1}(a)} \mu_A(x), \sup_{x\in f^{-1}(b)} \mu_A(x)\}\\ &= min\{\mu_{f(A)}(a), \mu_{f(A)}(b)\}\end{aligned}$$

and

$$\begin{aligned}\nu_{f(A)}(a*b) = \inf_{x\in f^{-1}(a*b)} \nu_A(x) &= \nu_A(x_0 * y_0)\\ &\leq \max\{\nu_A(x_0), \nu_A(y_0)\}\\ &= \max\{\inf_{x\in f^{-1}(b)} \nu_A(x), \inf_{x\in f^{-1}(b)} \nu_A(x)\}\\ &= \max\{\nu_{f(A)}(a), \nu_{f(A)}(b)\}\end{aligned}$$

Hence $f(A)$ is an intuitionistic fuzzy PMS-subalgebra of $Y$. □

THEOREM 3.2. *Let $f : X \to Y$ be a homomorphism of PMS-algebras. If $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy PMS-subalgebra of $Y$, then $f^{-1}(B)$ is an intuitionistic fuzzy PMS-subalgebra of $X$.*

PROOF: Assume that $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy PMS-subalgebra of $Y$ and let $x, y \in X$. Then,

$$\begin{aligned}\mu_{f^{-1}(B)}(x*y) = \mu_B(f(x*y)) &= \mu_B(f(x)*f(y))\\ &\geq \min\{\mu_B(f(x)), \mu_B(f(y))\}\\ &= \min\{\mu_{f^{-1}(B)}(x), \mu_{f^{-1}(B)}(y)\}\end{aligned}$$

and

$$\begin{aligned}\nu_{f^{-1}(B)}(x*y) = \nu_B(f(x*y)) &= \nu_B(f(x)*f(y))\\ &\leq \max\{\nu_B(f(x)), \nu_B(f(y))\}\\ &= \max\{\nu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(y)\}\end{aligned}$$

Therefore $f^{-1}(B)$ is an intuitionistic fuzzy PMS-subalgebra of $X$. □

The Converse of the above theorem is true if $f$ is a PMS-epimorphism.

THEOREM 3.3. *Let $f : X \to Y$ be an epimorphism of PMS-algebras and $B = (\mu_B, \nu_B)$ is a fuzzy set in $Y$. If $f^{-1}(B)$ is an intuitionistic fuzzy PMS-subalgebra of $X$, then $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy PMS-subalgebra of $Y$.*

PROOF: Assume that $f$ is an epimorphism of PMS-algebras and $f^{-1}(B)$ is an intuitionistic fuzzy PMS-subalgebra of $X$. Let $y_1, y_2 \in Y$. Since $f$ is an epimorphism of PMS-algebras, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Now,

$$\begin{aligned}
\mu_B(y_1 * y_2) &= \mu_B(f(x_1) * f(x_2)) \\
&= \mu_B(f(x_1 * x_2)) \\
&= \mu_{f^{-1}(B)}(x_1 * x_2) \\
&\geq \min\{\mu_{f^{-1}(B)}(x_1), \mu_{f^{-1}(B)}(x_2)\} \\
&= \min\{\mu_B(f(x_1)), \mu_B(f(x_2))\} \\
&= \min\{\mu_B(y_1), \mu_B(y_2)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_B(y_1 * y_2) &= \nu_B(f(x_1) * f(x_2)) \\
&= \nu_B(f(x_1 * x_2)) \\
&= \nu_{f^{-1}(B)}(x_1 * x_2) \\
&\leq \max\{\nu_{f^{-1}(B)}(x_1), \nu_{f^{-1}(B)}(x_2)\} \\
&= \max\{\nu_B(f(x_1)), \nu_B(f(x_2))\} \\
&= \max\{\nu_B(y_1), \nu_B(y_2)\}
\end{aligned}$$

Hence $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy PMS-Subalgebra of $Y$. □

DEFINITION 3.4. Let $f : X \to Y$ be a homomorphism of PMS-algebras for any intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ in $Y$. We define an intuitionistic fuzzy set $A^f = (\mu_A^f, \nu_A^f)$ in $X$ by $\mu_A^f(x) = \mu_A(f(x))$ and $\nu_A^f(x) = \nu_A(f(x)), \forall x \in X$.

In the next two theorems we characterize an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra using an intuitionistic fuzzy set defined above in Definition 3.4.

THEOREM 3.5. *Let $f : X \to Y$ be a homomorphism of PMS-algebras. If the intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $Y$, then the intuitionistic fuzzy set $A^f = (\mu_A^f, \nu_A^f)$ in $X$ is an intuitionistic fuzzy PMS-subalgebra of $X$.*

PROOF: Let $f$ be a homomorphism of PMS-algebras and let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $Y$. Let $x, y \in X$. Then

$$\begin{aligned} \mu_A^f(x * y) = \mu_A(f(x * y)) &= \mu_A(f(x) * f(y)) \\ &\geq \min\{\mu_A(f(x)), \mu_A(f(y))\} \\ &= \min\{\mu_A^f(x), \mu_A^f(y)\} \end{aligned}$$

and

$$\begin{aligned} \nu_A^f(x * y) = \nu_A(f(x * y)) &= \nu_A(f(x) * f(y)) \\ &\leq \max\{\nu_A(f(x)), \nu_A(f(y))\} \\ &= \max\{\nu_A^f(x), \nu_A^f(y)\} \end{aligned}$$

Hence $A^f = (\mu_A^f, \nu_A^f)$ is an intuitionistic fuzzy PMS-subalgebra of $X$. □

The Converse of Theorem 3.5 is also true if $f$ is an epimorphism of PMS-algebras as shown below in Theorem 3.6

THEOREM 3.6. *Let $f : X \to Y$ be an epimorphism of PMS-algebra. If $A^f = (\mu_A^f, \nu_A^f)$ is an intuitionistic fuzzy PMS-subalgebra of $X$, then $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $Y$.*

PROOF: Let $A^f = (\mu_A^f, \nu_A^f)$ be an intuitionistic fuzzy PMS-subalgebra in $X$ and let $x, y \in Y$. Then there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Now we have,

$$\begin{aligned} \mu_A(x * y) &= \mu_A(f(a) * f(b)) \\ &= \mu_A(f(a * b)) \\ &= \mu_A^f(a * b) \\ &\geq \min\{\mu_A^f(a), \mu_A^f(b)\} \end{aligned}$$

$$
\begin{aligned}
&= \min\{\mu_A(f(a)), \mu_A(f(b))\} \\
&= \min\{\mu_A(x), \mu_A(y)\}
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_A(x * y) &= \nu_A(f(a) * f(b)) \\
&= \nu_A(f(a * b)) \\
&= \nu_A^f(a * b) \\
&\leq \max\{\nu_A^f(a), \nu_A^f(b)\} \\
&= \max\{\nu_A(f(a)), \nu_A(f(b))\} \\
&= \max\{\nu_A(x), \nu_A(y)\}
\end{aligned}
$$

Hence $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $Y$. □

As a consequence of Theorems 3.5 and 3.6 we obtain the next theorem.

THEOREM 3.7. *Let $f : X \to Y$ be an epimorphism of PMS-algebra. Then $A^f = (\mu_A^f, \nu_A^f)$ is an intuitionistic fuzzy PMS-subalgebra of $X$ if and only if $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $Y$.*

## 4. Cartesian Product of Intuitionistic Fuzzy PMS-subalgebras

In this section, we discuss the concept of Cartesian product and the strongest fuzzy relation on intuitionistic fuzzy PMS-algebras. We prove that the Cartesian product of two intuitionistic fuzzy PMS-subalgebras is again an intuitionistic fuzzy PMS-subalgebra and some other results are also investigated.

LEMMA 4.1. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two intuitionistic fuzzy PMS-subalgebras of $X$ and $Y$ respectively. Then*

$$\mu_{A\times B}(0, 0) \geq \mu_{A\times B}(x, y)$$

*and*

$$\nu_{A\times B}(0, 0) \leq \nu_{A\times B}(x, y), \forall (x, y) \in X \times Y.$$

PROOF: Let $(x, y) \in X \times Y$. Then
$\mu_{A\times B}(0,0) = \min\{\mu_A(0), \mu_B(0)\} \geq \min\{\mu_A(x), \mu_B(y)\} = \mu_{A\times B}(x,y)$ and
$\nu_{A\times B}(0,0) = \max\{\nu_A(0), \nu_B(0)\} \leq \max\{\nu_A(x), \nu_B(y)\} = \nu_{A\times B}(x,y)$ □

THEOREM 4.2. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be any two intuitionistic fuzzy PMS-subalgebras of $X$ and $Y$ respectively. Then $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$.*

PROOF: Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned}\mu_{A\times B}((x_1,y_1)*(x_2,y_2)) &= \mu_{A\times B}(x_1*x_2,y_1*y_2)\\ &= \min\{\mu_A(x_1*x_2),\mu_B(y_1*y_2)\}\\ &\geq \min\{\min\{\mu_A(x_1),\mu_A(x_2)\},\min\{\mu_B(y_1),\mu_B(y_2)\}\}\\ &= \min\{\min\{\mu_A(x_1),\mu_B(y_1)\},\min\{\mu_A(x_2),\mu_B(y_2)\}\}\\ &= \min\{\mu_{A\times B}(x_1,y_1), \mu_{A\times B}(x_2,y_2)\}\end{aligned}$$

and

$$\begin{aligned}\nu_{A\times B}((x_1,y_1)*(x_2, y_2)) &= \nu_{A\times B}(x_1 * x_2,y_1 * y_2)\\ &= \max\{\nu_A(x_1*x_2),\nu_B(y_1*y_2)\}\\ &\leq \max\{\max\{\nu_A(x_1),\nu_A(x_2)\},\max\{\nu_B(y_1),\nu_B(y_2)\}\}\\ &= \max\{\max\{\nu_A(x_1),\nu_B(y_1)\},\max\{\nu_A(x_2),\nu_B(y_2)\}\}\\ &= \max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}\end{aligned}$$

Hence $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$. □

THEOREM 4.3. *Let $A$ and $B$ be intuitionistic fuzzy subsets of the PMS-algebras $X$ and $Y$ respectively. Suppose that $0$ and $0'$ are the constant elements of $X$ and $Y$ respectively. If $A \times B$ is an intuitionistic fuzzy PMS-subalgebras of $X \times Y$, then at least one of the following two statements holds.*

(i) $\mu_A(x) \leq \mu_B(0')$ *and* $\nu_A(x) \geq \nu_B(0')$,*for all* $x \in X$,

(ii) $\mu_B(y) \leq \mu_A(0)$ *and* $\nu_B(y) \geq \nu_A(0)$, *for all* $y \in Y$.

PROOF: Let $A \times B$ be an intuitionistic fuzzy PMS-subalgebra of $X \times Y$. Suppose that none of the statements $(i)$ and $(ii)$ holds. Then we can find $x \in X$ and $y \in Y$ such that $\mu_A(x) > \mu_B(0'), \nu_A(x) < \nu_B(0')$ and $\mu_B(y) > \mu_A(0), \nu_B(y) < \nu_A(0)$. Then we have

$$\mu_{A\times B}(x,y) = \min\{\mu_A(x), \mu_B(y)\} > \min\{\mu_B(0'), \mu_A(0)\} = \mu_{A\times B}(0,0')$$

and

$$\nu_{A\times B}(x,y) = \max\{\nu_A(x), \nu_B(y)\} < \max\{\nu_B(0'), \nu_A(0)\} = \nu_{A\times B}(0,0'),$$

which leads to

$$\mu_{A\times B}(x,y) > \mu_{A\times B}(0,0') \text{ and } \nu_{A\times B}(x,y) < \nu_{A\times B}(0,0').$$

This contradicts Lemma 4.1. Hence, either $(i)$ or $(ii)$ holds □

THEOREM 4.4. *Let $A$ and $B$ be intuitionistic fuzzy subsets of PMS-algebras $X$ and $Y$ respectively such that $\mu_A(x) \leq \mu_B(0')$ and $\nu_A(x) \geq \nu_B(0')$ for all $x \in X$, where $0'$ is a constant in $Y$. If $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$, then $A$ is an intuitionistic fuzzy PMS-subalgebra of $X$.*

PROOF: Let $x, y \in X$. Then $(x, 0'), (y, 0') \in X \times Y$. Since $\mu_A(x) \leq \mu_B(0')$ and $\nu_A(x) \geq \nu_B(0')$ for all $x \in X$, then for all $x, y \in X$ we get,

$$\begin{aligned}\mu_A(x*y) &= \min\{\mu_A(x*y), \mu_B(0'*0')\}\\ &= \mu_{A\times B}(x*y, 0'*0')\\ &= \mu_{A\times B}((x,0')*(y,0'))\\ &\geq \min\{\mu_{A\times B}(x,0'), \mu_{A\times B}(y,0')\}\\ &= \min\{\min\{\mu_A(x), \mu_B(0')\}, \min\{\mu_A(y), \mu_B(0')\}\}\\ &= \min\{\mu_A(x), \mu_A(y)\}\end{aligned}$$

and

$$\begin{aligned}\nu_A(x*y) &= \max\{\nu_A(x*y), \nu_B(0'*0')\}\\ &= \nu_{A\times B}(x*y, 0'*0')\\ &= \nu_{A\times B}((x,0')*(y,0'))\\ &\leq \max\{\nu_{A\times B}(x,0'), \nu_{A\times B}(y,0')\}\\ &= \max\{\max\{\nu_A(x), \nu_B(0')\}, \max\{\nu_A(y), \nu_B(0')\}\}\\ &= \max\{\nu_A(x), \nu_A(y)\}\end{aligned}$$

Hence $\mu_A(x*y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x*y) \leq \max\{\nu_A(x), \nu_A(y)\}$

Therefore $A$ is an intuitionistic fuzzy PMS-subalgebra of $X$. □

THEOREM 4.5. *Let $A$ and $B$ be intuitionistic fuzzy subsets of PMS-algebras $X$ and $Y$ respectively such that $\mu_B(y) \leq \mu_A(0)$ and $\nu_B(y) \geq \nu_A(0)$ for all $y \in Y$, where $0$ is a constant in $X$. If $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$, then $B$ is an intuitionistic fuzzy PMS-subalgebra of $Y$.*

PROOF: Let $x, y \in Y$. Then $(0, x), (0, y) \in X \times Y$. Since $\mu_B(y) \leq \mu_A(0)$ and $\nu_B(y) \geq \nu_A(0)$ for all $y \in Y$, then for all $x, y \in Y$ we get,

$$\begin{aligned}
\mu_B(x*y) &= \min\{\mu_A(0*0), \mu_B(x*y)\} \\
&= \mu_{A\times B}(0*0, x*y) \\
&= \mu_{A\times B}((0,x)*(0,y)) \\
&\geq \min\{\mu_{A\times B}(0,x), \mu_{A\times B}(0,y)\} \\
&= \min\{\min\{\mu_A(0), \mu_B(x)\}, \min\{\mu_A(0), \mu_B(y)\}\} \\
&= \min\{\mu_B(x), \mu_B(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\nu_B(x*y) &= \max\{\nu_A(0*0), \nu_B(x*y)\} \\
&= \nu_{A\times B}(0*0, x*y) \\
&= \nu_{A\times B}((0,x)*(0,y)) \\
&\leq \max\{\nu_{A\times B}(0,x), \nu_{A\times B}(0,y)\} \\
&= \max\{\max\{\nu_A(0), \nu_B(x)\}, \max\{\nu_A(0), \nu_B(y)\}\} \\
&= \max\{\nu_B(x), \nu_B(y)\}
\end{aligned}$$

Hence $\mu_B(x*y) \geq \min\{\mu_B(x), \mu_B(y)\}$ and $\nu_B(x*y) \leq \max\{\nu_B(x), \nu_B(y)\}$

Therefore $B$ is an intuitionistic fuzzy PMS-subalgebra of $Y$. □

From Theorems 4.3 , 4.4 and 4.5, we have the following:

COROLLARY 4.6. Let $A$ and $B$ be intuitionistic fuzzy subsets of PMS-algebras $X$ and $Y$ respectively. If $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$, then either $A$ is an intuitionistic fuzzy PMS-subalgebra of $X$ or $B$ is an intuitionistic fuzzy PMS-subalgebra of $Y$.

PROOF: Since $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$,

$$\mu_{A\times B}((x_1, y_1) * (x_2, y_2)) \geq \min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\} \tag{4.1}$$
$$\nu_{A\times B}((x_1, y_1) * (x_2, y_2)) \leq \max\{\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2)\} \tag{4.2}$$

If we put $x_1 = 0 = x_2$ in (4.1), we get

$$\begin{aligned}
&\mu_{A\times B}((0, y_1)*(0,y_2)) \geq \min\{\mu_{A\times B}(0, y_1),\mu_{A\times B}(0,y_2)\}\\
&\Rightarrow \mu_{A\times B}(0*0,y_1*y_2) \geq \min\{\mu_{A\times B}(0,y_1),\mu_{A\times B}(0,y_2)\}\\
&\Rightarrow \mu_{A\times B}(0,y_1*y_2) \geq \min\{\mu_{A\times B}(0,y_1),\mu_{A\times B}(0,y_2)\}\\
&\Rightarrow \min\{\mu_A(0),\mu_B(y_1*y_2)\} \geq \min\{\min\{\mu_A(0),\mu_B(y_1)\},\min\{\mu_A(0),\mu_B(y_2)\}\}
\end{aligned}$$

Hence, $\mu_B(y_1 * y_2) \geq \min\{\mu_B(y_1), \mu_B(y_2)\}$. Also,if we put $x_1 = 0 = x_2$ in (4.2), we get

$$\begin{aligned}
&\nu_{A\times B}((0,y_1)*(0, y_2)) \leq \max\{\nu_{A\times B}(0, y_1),\nu_{A\times B}(0,y_2)\}\\
&\Rightarrow \nu_{A\times B}(0*0, y_1*y_2) \leq \max\{\nu_{A\times B}(0,y_1),\nu_{A\times B}(0,y_2)\}\\
&\Rightarrow \nu_{A\times B}(0,y_1*y_2) \leq \max\{\nu_{A\times B}(0,y_1),\nu_{A\times B}(0,y_2)\}\\
&\Rightarrow \max\{\nu_A(0), \nu_B(y_1*y_2)\} \leq \max\{\max\{\nu_A(0),\nu_B(y_1)\},\max\{\nu_A(0),\nu_B(y_2)\}\}
\end{aligned}$$

Hence $\nu_B(y_1 * y_2) \leq \max\{\nu_B(y_1), \nu_B(y_2)\}$ and $B$ is an intuitionistic fuzzy PMS-subalgebra of $Y$.

Similarly, we prove that $A$ is an intuitionistic fuzzy PMS-subalgebra of $X$ by putting $y_1 = 0 = y_2$ in (4.1) and (4.2). □

THEOREM 4.7. *Let $A$ and $B$ be any intuitionistic fuzzy subsets of $X$ and $Y$ respectively. Then $A \times B$ is an intuitionistic fuzzy PMS-subalgebra of $X \times Y$ if and only if $\mu_{A\times B}$ and $\overline{\nu}_{A\times B}$ are fuzzy PMS-subalgebra of $X \times Y$, where $\overline{\nu}_{A\times B}$ is the complement of $\nu_{A\times B}$ .*

PROOF: Let $A \times B$ be an intuitionistic fuzzy PMS-subalgebra of $X \times Y$. Then by Definition 2.9 $\mu_{A\times B}((x_1, y_1) * (x_2, y_2)) \geq \min\{\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)\}$ and $\nu_{A\times B}((x_1, y_1) * (x_2, y_2)) \leq \max\{\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2)\}$ ,$\forall\,(x_1, y_1),(x_2, y_2) \in X \times Y$. Hence $\mu_{A\times B}$ is a fuzzy PMS-subalgebra of $X \times Y$ by Definition 2.5. Now for all $(x_1, y_1),(x_2, y_2) \in X \times Y$.

$$\begin{aligned}\overline{\nu}_{A\times B}((x_1,y_1)*(x_2,y_2)) &= 1-\nu_{A\times B}((x_1,y_1)*(x_2,y_2))\\ &\geq 1-\max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}\\ &= \min\{1-\nu_{A\times B}(x_1,y_1),1-\nu_{A\times B}(x_2,y_2)\}\\ &= \min\{\overline{\nu}_{A\times B}(x_1,y_1),\overline{\nu}_{A\times B}(x_2,y_2)\}\end{aligned}$$

Hence $\overline{\nu}_{A\times B}((x_1,y_1)*(x_2,y_2)) \geq \min\{\overline{\nu}_{A\times B}(x_1,y_1),\overline{\nu}_{A\times B}(x_2,y_2)\}$

Thus, $\overline{\nu}_{A\times B}$ is a fuzzy PMS-subalgebra of $X\times Y$.

Conversely, assume $\mu_{A\times B}$ and $\overline{\nu}_{A\times B}$ are fuzzy PMS-subalgebra of $X\times Y$. Then we have that $\mu_{A\times B}((x_1,y_1)*(x_2,y_2)) \geq \min\{\mu_{A\times B}(x_1,y_1), \mu_{A\times B}(x_2,y_2)\}$ and $\overline{\nu}_{A\times B}((x_1,y_1)*(x_2,y_2)) \geq \min\{\overline{\nu}_{A\times B}(x_1,y_1), \overline{\nu}_{A\times B}(x_2,y_2)\}$ for all $(x_1,y_1),(x_2,y_2)\in X\times Y$. So we need to show that $\nu_{A\times B}((x_1,y_1)*(x_2,y_2)) \leq \max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}$ for all $(x_1,y_1),(x_2,y_2)\in X\times Y$.

Now,

$$\begin{aligned}1-\nu_{A\times B}((x_1,y_1)*(x_2,y_2) &= \overline{\nu}_{A\times B}((x_1,y_1)*(x_2,y_2))\\ &\geq \min\{\overline{\nu}_{A\times B}(x_1,y_1),\overline{\nu}_{A\times B}(x_2,y_2)\}\\ &= \min\{1-\nu_{A\times B}(x_1,y_1),1-\nu_{A\times B}(x_2,y_2)\}\\ &= 1-\max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\},\end{aligned}$$

and so $\nu_{A\times B}((x_1,y_1)*(x_2,y_2) \leq \max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}$. Hence $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$. □

THEOREM 4.8. *Let $A$ and $B$ be any intuitionistic fuzzy subsets of $X$ and $Y$ respectively, then $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$ if and only if $\Box(A\times B)$ and $\Diamond(A\times B)$ are intuitionistic fuzzy PMS-subalgebra of $X\times Y$*

PROOF: Suppose $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$. Then $\mu_{A\times B}((x_1,y_1)*(x_2,y_2) \geq \min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}$ and $\nu_{A\times B}((x_1,y_1)*(x_2,y_2) \leq \max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}$, for all $(x_1,y_1),(x_2,y_2)\in X\times Y$

(i) To prove $\Box(A\times B)$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$, it suffices to show that for $(x_1,y_1),(x_2,y_2)\in X\times Y$, $\overline{\mu}_{A\times B}((x_1,y_1)*(x_2,y_2) \leq \min\{\overline{\mu}_{A\times B}(x_1,y_1),\overline{\mu}_{A\times B}(x_2,y_2)\}$. Now let $(x_1,y_1),(x_2,y_2)$ $\in X\times Y$

$$\begin{aligned}\overline{\mu}_{A\times B}((x_1,y_1)*(x_2,y_2)) &= 1-\mu_{A\times B}((x_1,y_1)*(x_2,y_2))\\ &\leq 1-\min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}\end{aligned}$$

$$= \max\{1-\mu_{A\times B}((x_1,y_1),1-\mu_{A\times B}(x_2,y_2))\}$$
$$= \max\{\overline{\mu}_{A\times B}((x_1,y_1),\overline{\mu}_{A\times B}(x_2,y_2))\},$$

whence $\overline{\mu}_{A\times B}((x_1,y_1)*(x_2,y_2)) \leq \max\{\overline{\mu}_{A\times B}(x_1,y_1),\overline{\mu}_{A\times B}(x_2,y_2)\}$ follows. Hence $\Box(A\times B)$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$.

(ii) To prove $\Diamond(A\times B)$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$, it suffices to show that $\overline{\nu}_{A\times B}((x_1,y_1)*(x_2,y_2) \geq \min\{\overline{\nu}_{A\times B}(x_1,y_1), \overline{\nu}_{A\times B}(x_2,y_2)\}$. Now let $(x_1,y_1),(x_2,y_2)\in X\times Y$ , then

$$\begin{aligned}\overline{\nu}_{A\times B}((x_1,y_1)*(x_2,y_2)) &= 1-\nu_{A\times B}((x_1,y_1)*(x_2,y_2))\\ &\geq 1-\max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}\\ &= \min\{1-\nu_{A\times B}((x_1,y_1),1-\nu_{A\times B}(x_2,y_2))\}\\ &= \min\{\overline{\nu}_{A\times B}((x_1,y_1),\overline{\nu}_{A\times B}(x_2,y_2))\},\end{aligned}$$

whence $\overline{\nu}_{A\times B}((x_1,y_1)*(x_2,y_2)) \geq \min\{\overline{\nu}_{A\times B}(x_1,y_1),\overline{\nu}_{A\times B}(x_2,y_2)\}$ follows. Hence $\Diamond(A\times B)$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$.

The proof of the converse is trivial. □

Definition 4.9. Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ are intuitionistic fuzzy subset of PMS-algebras $X$ and $Y$ reapectively. For $t,s\in[0,1]$ satisfying the condition $t+s\leq 1$, the set $U(\mu_{A\times B},t)=\{(x,y)\in X\times Y|\mu_{A\times B}(x,y)\geq t\}$ is called upper $t$-level set of $A\times B$ and the set $L(\nu_{A\times B},s)$ $=\{(x,y)\in X\times Y|\nu_{A\times B}(x,y)\leq s\}$ is called lower $s$-level set of $A\times B$.

Theorem 4.10. *Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be intuitionistic fuzzy subsets of $X$ and $Y$ reapectively. Then $A\times B$ is an intuitionistic fuzzy PMS-subalgebras of $X\times Y$ if and only if the nonempty upper t-level set $U(\mu_{A\times B},t)$ and the nonempty lower s-level set $L(\nu_{A\times B},s)$ are PMS-subalgebras of $X\times Y$ for any $t,s\in[0,1]$ with $t+s\leq 1$.*

Proof: Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be intuitionistic fuzzy subsets of $X$ and $Y$ respectively. Let $(x_1,y_1),(x_2,y_2)\in X\times Y$ such that $(x_1,y_1),(x_2,y_2)\in U(\mu_{A\times B},t)$ for $t\in[0,1]$. Then $\mu_{A\times B}(x_1,y_1)\geq t$ and $\mu_{A\times B}(x_2,y_2)\geq t$. Since $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$, we have

$$\begin{aligned}\mu_{A\times B}((x_1,y_1)*(x_2,y_2)) &\geq \min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}\\ &\geq \min\{t,t\}=t\end{aligned}$$

Therefore, $(x_1,y_1)*(x_2,y_2)\in U(\mu_{A\times B},t)$. Hence $U(\mu_{A\times B},t)$ is a PMS-subalgebra of $X\times Y$.

Also, Let $(x_1,y_1),(x_2,y_2)\in X\times Y$ such that $(x_1,y_1),(x_2,y_2)\in L(\nu_{A\times B},s)$ for $s\in[0,1]$. Then $\nu_{A\times B}(x_1,y_1)\leq s$ and $\nu_{A\times B}(x_2,y_2)\leq s$. Since $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$, we have

$$\begin{aligned}\nu_{A\times B}((x_1,y_1)*(x_2,y_2)) &\leq \max\{\nu_{A\times B}(x_1,y_1),\nu_{A\times B}(x_2,y_2)\}\\ &\leq \max\{s,s\}=s\end{aligned}$$

Therefore, $(x_1,y_1)*(x_2,y_2)\in L(\nu_{A\times B},s)$. Hence $L(\nu_{A\times B},s)$ is a PMS-subalgebra of $X\times Y$.

Conversely, Suppose $U(\mu_{A\times B},t)$ and $L(\nu_{A\times B},s)$ are PMS-subalgebra of $X\times Y$ for any $t,s\in[0,1]$ with $t+s\leq 1$. Assume that $A\times B$ is not an intuitionistic fuzzy PMS-subalgebra of $X\times Y$. Then there exist $(x_1,y_1),(x_2,y_2)\in X\times Y$ such that

$$\mu_{A\times B}((x_1,y_1)*(x_2,y_2))<\min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}.$$

Then by taking $t_0=\frac{1}{2}\{\mu_{A\times B}((x_1,y_1)*(x_2,y_2))+\min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}\}$, we get $\mu_{A\times B}((x_1,y_1)*(x_2,y_2))<t_0<\min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}$. Hence, $(x_1,y_1)*(x_2,y_2)\notin U(\mu_{A\times B},t_0)$ but $(x_1,y_1)\in U(\mu_{A\times B},t_0)$ and $(x_2,y_2)\in U(\mu_{A\times B},t_0)$, This implies $U(\mu_{A\times B},t_0)$ is not a PMS-subalgebra of $X\times Y$, which is a contradiction. Therefore $\mu_{A\times B}((x_1,y_1)*(x_2,y_2))\geq\min\{\mu_{A\times B}(x_1,y_1),\mu_{A\times B}(x_2,y_2)\}$.
Similarly, $\nu_{A\times B}((x_1,y_1)*(x_2,y_2))\leq\max\{\mu_{A\times B}(x_1,y_y),\mu_{A\times B}(x_2,y_2)\}$.

Hence $A\times B$ is an intuitionistic fuzzy PMS-subalgebra of $X\times Y$. □

THEOREM 4.11. *Let $A=(\mu_A,\nu_A)$ be an intuitionistic fuzzy subset of PMS-algebra $X$ and let $R_A$ be the strongest intutionistic fuzzy PMS-relation on $X$. If $R_A$ is an intuitionistic fuzzy PMS-subalgebra of $X\times X$, then $\mu_A(0)\geq\mu_A(x)$ and $\nu_A(0)\leq\nu_A(x)$, for all $x\in X$.*

PROOF: Since $R_A$ is an intuitionistic fuzzy PMS-subalgebra of $X\times X$, it follows from Lemma 4.1 that $\mu_{R_A}(0,0)\geq\mu_{R_A}(x,x)$ and $\nu_{R_A}(0,0)\leq\nu_{R_A}(x,x)$. Then, we have $\min\{\mu_A(0),\mu_A(0)\}=\mu_{R_A}(0,0)\geq\mu_{R_A}(x,x)=$

$\min\{\mu_A(x), \mu_A(x)\}$, where $(0,0) \in X \times X$, which implies $\min\{\mu_A(0), \mu_A(0)\} \geq \min\{\mu_A(x), \mu_A(x)\}$, and so, $\mu_A(0) = \min\{\mu_A(0), \mu_A(0)\} \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$. Moreover, $\max\{\nu_A(0), \nu_A(0)\} = \nu_{R_A}(0,0) \leq \nu_{R_A}(x,x) = \max\{\nu_A(x), \nu_A(x)\}$, where $(0,0) \in X \times X$, whence follows $\max\{\nu_A(0), \nu_A(0)\} \leq \max\{\nu_A(x), \nu_A(x)\}$ and further $\nu_A(0) = \max\{\nu_A(0), \nu_A(0)\} \leq \max\{\nu_A(x), \nu_A(x)\} = \nu_A(x)$.

Hence $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$, for all $x \in X$. □

THEOREM 4.12. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of a PMS-algebra $X$ and let $R_A$ be the strongest intuitionistic fuzzy PMS-relation on $X$. Then $A$ is an intuitionistic fuzzy PMS-subalgebra of $X$ if and only if $R_A$ is an intuitionistic fuzzy PMS-subalgebra of $X \times X$.*

PROOF: Assume that $A$ is an intuitionistic fuzzy PMS-subalgebra $X$. Let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then, we have

$$\begin{aligned}
\mu_{R_A}((x_1, x_2)*(y_1, y_2)) &= \mu_{R_A}(x_1*y_1, x_2*y_2) \\
&= \min\{\mu_A(x_1*y_1), \mu_A(x_2*y_2)\} \\
&\geq \min\{\min\{\mu_A(x_1), \mu_A(y_1)\}, \min\{\mu_A(x_2), \mu_A(y_2)\}\} \\
&= \min\{\min\{\mu_A(x_1), \mu_A(x_2)\}, \min\{\mu_A(y_1), \mu_A(y_2)\}\} \\
&= \min\{\mu_{R_A}(x_1, x_2), \mu_{R_A}(y_1, y_2)\}.
\end{aligned}$$

and

$$\begin{aligned}
\nu_{R_A}((x_1, x_2)*(y_1, y_2)) &= \nu_{R_A}(x_1*y_1, x_2*y_2) \\
&= \max\{\nu_A(x_1*y_1), \nu_A(x_2*y_2)\} \\
&\leq \max\{\max\{\nu_A(x_1), \nu_A(y_1)\}, \max\{\nu_A(x_2), \nu_A(y_2)\}\} \\
&= \max\{\max\{\nu_A(x_1), \nu_A(x_2)\}, \max\{\nu_A(y_1), \nu_A(y_2)\}\} \\
&= \max\{\nu_{R_A}(x_1, x_2), \nu_{R_A}(y_1, y_2)\}.
\end{aligned}$$

Hence $R_A$ is an intuitionistic fuzzy PMS-subalgebra of $X \times X$.

Conversely, assume $R_A$ is an intuitionistic fuzzy PMS-subalgebra of $X \times X$. Let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then

$$\begin{aligned}
\min\{\mu_A(x_1*y_1), \mu_A(x_2*y_2)\} &= \mu_{R_A}(x_1*y_1, x_2*y_2) \\
&= \mu_{R_A}((x_1, x_2)*(y_1, y_2)) \\
&\geq \min\{\mu_{R_A}(x_1, x_2), \mu_{R_A}(y_1, y_2)\} \\
&= \min\{\min\{\mu_A(x_1), \mu_A(x_2)\}, \\
&\qquad \min\{\mu_A(y_1), \mu_A(y_2)\}\}
\end{aligned}$$

In particular, if we take, $x_2 = y_2 = 0$ (or respectively $x_1 = y_1 = 0$), then we get $\mu_A(x_1 * y_1) \geq min\{\mu_A(x_1), \mu_A(y_1)\}$ (or resp. $\mu_A(x_2 * y_2) \geq \min\{\mu_A(x_2), \mu_A(y_2)\}$)and

$$\begin{aligned}
\max\{\nu_A(x_1*y_1), \nu_A(x_2*y_2)\} &= \nu_{R_A}(x_1*y_1, x_2*y_2) \\
&= \nu_{R_A}(x_1, x_2)*(y_1, y_2) \\
&\leq \max\{\nu_{R_A}(x_1, x_2), \nu_{R_A}(y_1, y_2)\} \\
&= \max\{\max\{\nu_A(x_1), \nu_A(x_2)\}, \\
&\qquad \max\{\nu_A(y_1), \nu_A(y_2)\}\}
\end{aligned}$$

In particular, if we take, $x_2 = y_2 = 0$ (or respectively $x_1 = y_1 = 0$), then we get $\nu_A(x_1 * y_1) \leq \max\{\nu_A(x_1), \nu_B(y_1)\}$ (or resp. $\nu_A(x_1 * y_1) \leq \max\{\mu_A(x_1), \mu_A(y_1)\}$)

Therefore $A$ is an intuitionistic fuzzy PMS-subalgebra of $X$ □

## 5. Conclusion

In this paper, we discussed the concept of intuitionistic fuzzy PMS-subalgebra under homomorphism and Cartesian product in a PMS-algebra. We confirmed that the homomorphic image and the homomorphic inverse image of an intuitionistic fuzzy PMS-subalgebra in a PMS-algebra are intuitionistic fuzzy PMS-subalgebras. We also proved that the Cartesian product of the intuitionistic fuzzy PMS-subalgebras of a PMS-algebra is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra. Furthermore, we characterized the Cartesian products of intuitionistic fuzzy PMS-subalgebras in terms of their level sets. Finally, we discussed the concept of the strongest intuitionistic fuzzy PMS-relation on an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra and investigated some of its properties. We will further extend these concepts to intuitionistic fuzzy PMS-ideals of a PMS-algebra for new results in our future work.

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