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BASIC FOUR-VALUED SYSTEMS OF CYCLIC NEGATIONS[1]

Abstract

We consider an example of four valued semantics partially inspired by quantum computations and negation-like operations occurred therein. In particular we consider a representation of so called square root of negation within this four valued semantics as an operation which acts like a cycling negation. We define two variants of logical matrices performing different orders over the set of truth values. Purely formal logical result of our study consists in axiomatizing the logics of defined matrices as the systems of binary consequence relation and proving correctness and completeness theorems for these deductive systems.

Keywords: Generalized truth values, consequence relation, first degree entailment.

1. Introduction

The study of properties of negation-like connectives constitutes nowadays is a well established area of interdisciplinary research activity, including purely logical investigations (consult collective monographs [13, 26]). Negation often expresses the characteristic features of logical systems acting

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thereby as a mean for distinguishing and systematizing them (see, for instance, [16] for the treatment of different types of paraconsistent logics in accordance with the properties of negations introduced there). In the literature one can find examples of hierarchic structures over the sets of negations, aimed to reflect their logical properties. Probably the best known is the "kite of negations" proposed by M. Dunn in [8] and refined in the subsequent articles.

In this paper, we are not intent on providing a complete picture of some big family of negation-like operations, instead we concentrate on a particular type of negation which may be characterized as a *cyclic* operation over certain set of truth values. Specifically we are interested in its behaviour in the context of four-valued semantics, the breeding ground of many well known non-classical logics.

Occasionally our research was brought to life with an interest to the problematics of quantum computation and its possible representations within the semantic framework of non-classical logic. In particular the reflections on one of the most unusual quantum gates, the square root of negation, induced a unary operation on the four-element set of truth values. On the syntactic level, we defined two logical systems considerably differing from each other with respect to the set of deductive postulates but sharing "classicality" of double negation. This particular feature is inherent in some other non-classical logics [14, 15, 18, 28, 29].

2. Cyclic negation in the generalized truth values setting

Our interest to studies of cyclic negation stems from the different sources. This kind of negation is primarily known in the field of Post algebras and their logics (see [20, 21]). Another origin can be found in the context of four valued semantics and corresponding logics. According to [17], the first appearance of a cyclic negation in four valued framework can be found in [22], while [17] itself deals with the property of functional completeness for the expansions of Belnap-Dunn logic. In particular Belnap-Dunn logic equipped with cyclic negation in [22] is proved to be functionally complete. In [28], two versions of cycling negation appeared under the names *left* and *right turns* as the specific operations over the set of two-component generalized truth values[2], but they had not been studied there at any extent. Four-valued systems with some relatives of cyclic negation (different from ours) are investigated in [15, 18, 19]. One of the features of the negation-like operations studied there consists in their ability to simulate the properties of classical (and is some cases intuitionistic) negation via composition. It is worth noting that [14] addresses the problem of simulating conventional negations via other unary operations touching upon a cyclic negation.

2.1. The Basics of generalized truth values

The truth values that we concern with throughout this paper can be understood as a kind of generalized truth values. Although we start with the idea of how these truth values arise from the representation of quantum computational logic gates in the framework of four-valued semantics, later we show that the values can be generated in a regular way via elementary set-theoretical operations. Let us discuss this the process of introducing generalized truth values in more details.

Generalized truth values are the result of power-setting (or sometimes taking Cartesian product) of an initial set of truth-values. For example, if we start with the set $\mathbf{2}$ of classical values $\{t, f\}$, then the first stage of its generalization is the set $\mathscr{P}(\mathbf{2}) = \mathbf{4} = \{\{t, f\}, \{t\}, \{f\}, \varnothing\}$. Ordered by "definiteness-of-truth relation", the set $\mathbf{4}$ forms a well known lattice $\mathcal{FOUR}_2$ of Belnap's truth values (assuming that $\mathbf{T} = \{t\}$, $\mathbf{B} = \{t, f\}$, $\mathbf{F} = \{f\}$ and $\mathbf{N} = \varnothing$). This structure can also be considered as bilattice when the second, informational order, is taken into account (see Figure 1).

To proceed further, one needs to generalize a valuation function as well, to be a map from the set of propositional variables to the set $\mathbf{4}$. If we in a natural way extend valuation to arbitrary formula and define an appropriate consequence relation, we arrive at certain semantic logic. Interestingly, a logic whose consequence relation is defined via the logical ordering is exactly the useful 4-valued relevant logic constructed by [9] and [1, 2].

Generalization procedure has no limits. From $\mathbf{2}$, it leads through $\mathbf{4}$ to $\mathrm{P}(\mathbf{4})=\mathbf{16}$ and the trilattice $\mathcal{SIXTEEN}_3$ with three independent orderings. This algebraic structure is a special case of multilattice proposed and discussed in [23]. Moreover, two of these three ordering relations generate

[2] For the detailed account of this kind of compound truth values see [29].

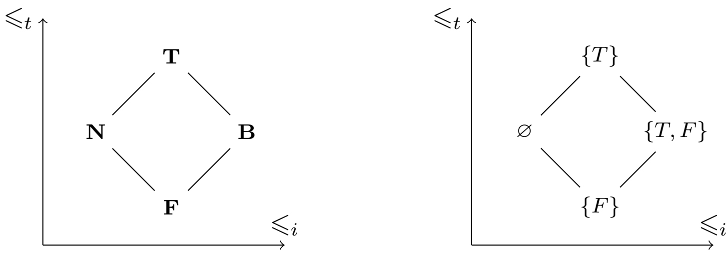

Figure 1. Bilattice $\mathcal{FOUR}_2$ in Belnap's and generalized truth-values setting.

useful 16-valued logics of the first-degree entailment [24]. If one takes the set **3** of strong Kleene's three-valued logic, it gives rise to a valuational system corresponding to the lattice $\mathcal{EIGHT}_3$ with three orderings [27]. And again this valuational structure generates the first-degree relevant logic.

Some constructions of the generalized truth valued might deviate from the paradigm pictured above. For example the values used in [29] are generated from the set $\{t, 1\}$ of two different types of truth, while false (of a certain type) is rendered as just the absence of truth (of the same type).

2.2. Four-valuedness and cyclic negation from quantum computations

Although this paper *does not* concern with quantum computations or their logic at all, some concepts from the field of quantum computational logic have inspired the four-valued semantics underlying the logics discussed below and, specifically, the choice of the unary operation acting over there. This section clarifies the origins of the family of truth values used below.

One of the ideas that motivated this paper, namely, to merge generalized truth values approach and quantum computation in a joint logical framework, was prompted by seminal writings of prominent logicians of past and present, and after all is connected with the search of answers to the question, what (modern) logic is.

The first one was proposed by G. Frege and J. Łukasiewicz many years ago and now enjoys a new lease on life within the project of generalized truth values. The core idea may be expressed in Łukasiewicz's words

– logic is the science of objects of a special kind, namely a science of logical values. Though seems strange, this understanding of logic is coherent with standard conception of logic, because the search for criteria of correct reasoning and argument immediately leads one to truth-(or, designated value-) preserving interpretation of logical inference.

Another conception of logic is due to J. van Benthem, who in [25] develops a program of Logical Dynamics, which presupposes the interpretation of logic as a theory of information-driven agency, being thus the study of explicit informational processes (inference, observation, communication). The latter interpretation may be seen as the other side of the same coin – in words of J. van Benthem, "inference is just one way of producing information, at best on a par, even for logic itself, with others" [3, p. 183], so it is little wonder that "inference and information update are intertwined" [3, p. 189].

One step away from here and just a moment to go, there is an idea to consider quantum logic as logic of quantum computation, where the latter offers a new possibility opened up by quantum gates to deal with information processing procedures being generalizations of reasoning and argument. An additional interest is connected with logical formalization of so called genuine quantum gates "that transform classical registers into quregisters that are superpositions: the square root of the negation and the square root of the identity" [5, p. 298]. According to [6] "logicians are now entitled to propose a new logical operation $\sqrt{\mathtt{NOT}}$. Why? Because a faithful physical model for it exists in nature".

Let us remind some key concepts of quantum computational logic (for more details see, for example, [4]). The unit of representation of quantum information is a qubit (from English "quantum bit"), $a|0\rangle + b|1\rangle$, where $|0\rangle$ and $|1\rangle$ are vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, respectively, written in so called Dirac notation, while a and b are complex numbers, the *amplitudes*, expressing the probabilities.

Quantum computational logic offers a broad family of operators, quantum logic gates[3], which in some cases can be rendered as the counterparts of classical logic gates and thus give rise to a family of propositional con-

[3]Well known examples are CNOT, TOFFOLI, FREDKIN, SWAP gates which perform reversible computation using some qubits as control registers for governing the actions on target bit. For example, CNOT negates its target bit if and only if the control bit is recognized as 1.

nectives in formal languages of quantum logical systems. But quantum computations provide also examples of non-classical gates. The square root of negation is of the special interest for us. For a qubit $|\varphi\rangle = a|0\rangle + b|1\rangle$, $\sqrt{\texttt{NOT}}(|\varphi\rangle) = \frac{1}{2}[(1+i)a + (1-i)b]|0\rangle + \frac{1}{2}[(1-i)a + (1+i)b]|1\rangle$, where i is an imaginary unit. While `NOT` gate transforms $|1\rangle$ into $|0\rangle$ and vice versa, $\sqrt{\texttt{NOT}}$ does only half of the work.

The key observation here is that the square root of the negation is a kind of "connective with memory". In particular, when applied twice to Truth, it returns Falsity and vice versa. At the same time, the first application to True or False gives intermediate value. Thus, to understand where to go after the first application of the square root of the negation, one should somehow remember the point of departure. The complex nature of generalized truth values allows to yield this peculiarity by preserving the component of the initial value. For example, starting with $\mathbf{T}$, the first application of the square root of the negation "adds" uncertainty thus producing $\mathbf{TU}$; the second application transforms it to $\mathbf{F}$; the third again adds $\mathbf{U}$ to $\mathbf{F}$ resulting in $\mathbf{FU}$; and finally after the fourth application we arrive at $\mathbf{T}$. So we can see that our representation of the square root of negation within four-valued framework is nothing more then *a cyclic negation*.

Thus we have new set of truth values, $\{\mathbf{T}, \mathbf{TU}, \mathbf{FU}, \mathbf{F}\}$, and an open choice of order relation and subset of the designated values. Below we consider two natural variants of partial order over this set with the same two-element subset of designated values, $\{\mathbf{T}, \mathbf{TU}\}$. The choice of this subset seems reasonable for several reasons. It contains Truth itself ($\mathbf{T}$) and the the other value ($\mathbf{TU}$), having something that we would call *a trace of truth*. Moreover, this subset is one of the two prime filters in lattice $4\mathcal{Q}$ described below.

In this paper, we consider two propositional logics, $\mathbf{CNL_4^2}$ and $\mathbf{CNLL_4^2}$, determined by four-valued matrices (with two-valued matrix filters) constructed over the set of generalized truth values inspired by quantum computations as explained above. Though these logics have much in common, they differ essentially with respect to the properties of negations and their interrelation with conjunction and disjunction.

2.3. Four-valued matrices

For both logics, $\mathbf{CNL_4^2}$ and $\mathbf{CNLL_4^2}$, we subsume the same propositional language $\mathcal{L}_{cnl}$ of the signature $\{\wedge, \vee, \neg\}$ over denumerable set of variables $\mathtt{Var}$ with the set of complex formulas $\mathtt{For}$ constructed according to the standard inductive definition.

On the basis of the set $\mathscr{U} = \{\mathbf{T}, \mathbf{TU}, \mathbf{FU}, \mathbf{F}\}$ we define two distinct matrices, $\mathcal{M}^{\mathbf{CNL_4^2}}$ and $\mathcal{M}^{\mathbf{CNLL_4^2}}$, over this set with the same subset of designated values $\mathcal{D} = \{\mathbf{T}, \mathbf{TU}\}$ and the same definition of unary operation $\mathcal{O} = \{\sim, \wedge, \vee\}$ differing with respect to meet and join in the lattice reducts of these matrices.

Tableau definitions for the binary operations $\wedge$ and $\vee$ can be easily imported from the order relations over the set of truth values represented via Hasse diagrams, depicted in Figure 2. Evidently these ordered sets of truth values constitute two simple lattices, $4\mathcal{Q}$ (left diagram) and $4\mathcal{LQ}$.

Definition 2.1. $\mathcal{M}^{\mathbf{CNL_4^2}}$ matrix is a structure $\langle \mathscr{U}, \{f_c\}_{c\in\mathcal{O}}, \mathcal{D}\rangle$, where the operations $f_\wedge$ and $f_\vee$ are defined as meet and join in $4\mathcal{Q}$, $f_\sim$ is defined via the following table:

x	$f_\sim(x)$
$\mathbf{T}$	$\mathbf{TU}$
$\mathbf{TU}$	$\mathbf{F}$
$\mathbf{F}$	$\mathbf{FU}$
$\mathbf{FU}$	$\mathbf{T}$

Definition 2.2. $\mathcal{M}^{\mathbf{CNLL_4^2}}$ matrix is a structure $\langle \mathscr{U}, \{g_c\}_{c\in\mathcal{O}}, \mathcal{D}\rangle$, where the operations $g_\wedge$ and $g_\vee$ are defined as meet and join in $4\mathcal{LQ}$, $g_\sim$ is defined via the same table as $f_\sim$.

A valuation v is a mapping $\mathtt{Var} \mapsto \mathscr{U}$. An extension of v to the set $\mathtt{For}$ depends on a matrix assumed. For example, in case of $\mathcal{M}^{\mathbf{CNL_4^2}}$ we define extension v_2 of v via following expressions for all $A, B \in \mathtt{For}$: $v_2(A \wedge B) = f_\wedge(v_2(A), v_2(B))$, $v_2(A \vee B) = f_\vee(v_2(A), v_2(B))$, $v_2(\neg A) = f_\sim(v_2(A))$. In the same manner we define an extension v_3 of a valuation over $\mathbf{CNLL_4^2}$ matrix, using operations $g_\sim$, $g_\wedge$ and $g_\vee$.

The semantic consequence relation is defined via preservation of a designated truth value and again relies on a matrix assumed:

Definition 2.3. For all $A, B \in \mathtt{For}$,

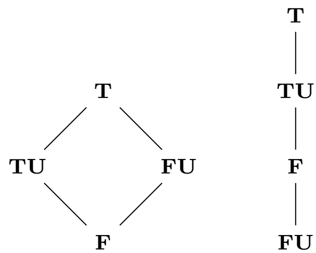

Figure 2. Lattices $4\mathcal{Q}$ and $4\mathcal{LQ}$.

(1) $A \vDash_{\mathbf{CNL_4^2}} B \iff v(A) \in \mathcal{D} \Rightarrow v(B) \in \mathcal{D}$, for each $\mathbf{CNL_4^2}$-valuation v,

(2) $A \vDash_{\mathbf{CNLL_4^2}} B \iff v(A) \in \mathcal{D} \Rightarrow v(B) \in \mathcal{D}$, for each $\mathbf{CNLL_4^2}$-valuation v.

It is instructive to examine set $\mathscr{U}$ from the generalized truth values perspective. A common way to construct a set of generalized truth values is to get powerset over some semantic basis. So, let us choose the basic set $\{\mathbf{T}, \mathbf{U}\}$, consisting of Truth and Uncertainty values, obtaining thereby the set of generalized truth values $\{\{\mathbf{T}, \mathbf{U}\}, \{\mathbf{T}\}, \{\mathbf{U}\}, \varnothing\}$. It is natural to think of $\{\mathbf{T}\}$ as just $\mathbf{T}$, while $\{\mathbf{T}, \mathbf{U}\}$ as our $\mathbf{TU}$. Then $\mathbf{U}$ is just "uncertainty without being true". Recall that the absence of truth can be understood as just being false. This suggests that $\mathbf{U}$ can be thought as $\mathbf{FU}$; likewise $\varnothing$ is just $\mathbf{F}$.

3. Binary consequence systems for $\mathbf{CNL_4^2}$ and $\mathbf{CNLL_4^2}$

To formalize semantically defined consequence relation we will use a specific variant of a logical calculus, "a binary consequence system"[4], which is typical of all **FDE**-related logics. The term "binary" means that a sequent[5]

[4] See [10, Chapter 6] for a discussion of terminology concerning to different presentations of logical systems. In particular our approach is called "binary implicational system" there.

[5] We use the term 'sequent' in a broad sense, not reffering here to the apparatus of Gentzen calculi.

is an expression of a form $A \vdash B$ which contains exactly one formula in the antecedent or consequent position. We take some *schemata* of sequents regarded as the axiomatic schemata. A sequent is an axiom if it is a particular instance of a schema. To make the presentation succinct we abbreviate $\sim\sim$ as $\sim^2$, $\sim\sim\sim$ as $\sim^3$ and so on.

DEFINITION 3.1. A sequent $A \vdash B$ is called $\mathbf{CNL^2_4}$-valid ($\mathbf{CNLL^2_4}$-valid) $\Longleftrightarrow$

$$A \vDash_{\mathbf{CNL^2_4}} B \quad (A \vDash_{\mathbf{CNLL^2_4}} B).$$

DEFINITION 3.2. A $\mathbf{CNL^2_4}$-proof (a $\mathbf{CNLL^2_4}$-proof) as a list of sequents each of them is whether an axiom of $\mathbf{CNL^2_4}$(an axiom of $\mathbf{CNLL^2_4}$) or derived from the previous items of the list using some rule of inference. A $\mathbf{CNL^2_4}$-proof ($\mathbf{CNLL^2_4}$-proof) for a sequent $A \vdash B$ is a $\mathbf{CNL^2_4}$-proof ($\mathbf{CNLL^2_4}$-proof) the last item of which coincides with $A \vdash B$. A sequent $A \vdash B$ is called $\mathbf{CNL^2_4}$-provable ($\mathbf{CNLL^2_4}$-provable) if there is a $\mathbf{CNL^2_4}$-proof ($\mathbf{CNLL^2_4}$-proof) for $A \vdash B$.

To indicate that a sequent $A \vdash B$ is $\mathbf{CNL^2_4}$-provable ($\mathbf{CNLL^2_4}$-provable) we also adopt the expression $A \vdash_{\mathbf{CNL^2_4}} B$ ($A \vdash_{\mathbf{CNLL^2_4}} B$).

$\mathbf{CNL^2_4}$ & $\mathbf{CNLL^2_4}$ COMMON AXIOMATIC SCHEMATA AND RULES OF INFERENCE:

(a1) $A \wedge B \vdash A$,
(a2) $A \wedge B \vdash B$,
(a3) $B \vdash A \vee B$,
(a4) $A \vdash A \vee B$,
(a5) $\sim A \wedge \sim B \vdash \sim(A \wedge B)$,
(a6) $\sim(A \vee B) \vdash \sim A \vee \sim B$,
(a7) $A \wedge \sim^2 A \vdash B$,
(a8) $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$
(a9) $A \vdash \sim^4 A$,
(a10) $\sim^4 A \vdash A$.

(r1) $A \vdash B,\ B \vdash C\ /\ A \vdash C$,
(r2) $A \vdash B,\ A \vdash C\ /\ A \vdash B \wedge C$,
(r3) $A \vdash C,\ B \vdash C\ /\ A \vee B \vdash C$,
(r4) $A \vdash B\ /\ \sim^2 B \vdash \sim^2 A$.

$\mathbf{CNL_4^2}$ ADDITIONAL AXIOMATIC SCHEMATA:

(b1) $\sim(A \wedge B) \vdash \sim A \wedge \sim B$, (b2) $\sim A \vee \sim B \vdash \sim(A \vee B)$.

$\mathbf{CNLL_4^2}$ ADDITIONAL AXIOMATIC SCHEMATA:

(c1) $\sim A \wedge \sim B \vdash \sim(A \vee B)$,
(c2) $\sim(A \wedge B) \vdash \sim A \vee \sim B$,
(c3) $\sim A \wedge \sim^2 A \vdash \sim(A \wedge B)$
(c4) $A \wedge \sim A \vdash \sim(A \vee B)$,
(c5) $\sim(A \vee B) \vdash \sim A \vee B$,
(c6) $\sim(A \vee B) \vdash \sim(B \vee A)$,
(c7) $\sim(A \wedge B) \vdash \sim(B \wedge A)$,
(c8) $(\sim(A \vee B) \wedge \sim(A \wedge B)) \vdash \sim A \wedge \sim B$.

PROPOSITION 3.3. The following sequents are provable in $\mathbf{CNL_4^2}$:

(1) $\sim A \wedge \sim B \vdash \sim(A \vee B)$,
(2) $\sim(A \wedge B) \vdash \sim A \vee \sim B$.

PROPOSITION 3.4. The following sequents are provable in both $\mathbf{CNL_4^2}$ and $\mathbf{CNLL_4^2}$.

(Id) $A \vdash A$
(De1) $\sim^2 A \wedge \sim^2 B \dashv\vdash \sim^2(A \vee B)$,
(De2) $\sim^2 A \vee \sim^2 B \dashv\vdash \sim^2(A \wedge B)$,
(T) $B \vdash A \vee \sim^2 A$.

PROOF: Let us show the proof for (T) only:

1. $A \wedge \sim^2 A \vdash \sim^2 B$ (a7)
2. $\sim^4 B \vdash \sim^2(A \wedge \sim^2 A)$ 1, (r4)
3. $\sim^2(A \wedge \sim^2 A) \vdash \sim^2 A \vee \sim^4 A$ (De2)
4. $\sim^2 A \vee \sim^4 A \vdash A \vee \sim^2 A$ (Id), (a3), (a4), (a10), (r1), (r3)
5. $B \vdash \sim^4 B$ (a9)
6. $B \vdash A \vee \sim^2 A$ 2, 3, 4, 5, (r1)

□

4. Systems of cyclic negation and classical logic

Systems $\mathbf{CNL_4^2}$ and $\mathbf{CNLL_4^2}$ have much in common with classical logic. Indeed, if we were intended to represent classical logic as a binary consequence system, we would take (a1)–(a4), (a8)–(a10) and (r1)–(r4), adding paradoxical postulates like (a7) (then, of course, a pair $\sim\sim$ should be treated as classical $\neg$). Is is well known that an alternative formulation of classical system is obtained by replacing contraposition rule with a full collection of De Morgan laws (but then both $A \wedge \neg A \vdash_{\mathbf{Cl}} B$ and $A \vdash_{\mathbf{Cl}} B \vee \neg B$ are needed, where $\vdash_{\mathbf{Cl}}$ stands for classical binary consequence relation) as axiomatic schemas. For further references we will denote this system as **Cl**.

As mentioned above, double $\sim$ have all these features of classical negation. Thus a kind of *intrinsic* classicality present in both our systems. More precisely we can represent this fact via translation function Φ from the language of classical logic $\mathcal{L}_{cl}$ (over the signature $\{\wedge, \vee, \neg\}$, with the set of formulas denoted as $\mathtt{For}_{cl}$) to the language of the present systems (with the proviso that both languages share the same denumerable set of propositional variables $\mathtt{Var} = \{p_1, p_2, \ldots\}$):

$$\begin{aligned} \Phi(p) &= p, \quad p \in \mathtt{Var}, \\ \Phi(A \circ B) &= \Phi(A) \circ \Phi(B), \quad \circ \in \{\wedge, \vee\}, \\ \Phi(\neg A) &= \sim\sim\Phi(A), \quad A, B \in \mathtt{For}_{cl}. \end{aligned}$$

We would like to show, that Φ is not only a translation, but an embedding function as well[6]. We prove this statement via semantic argument. Let us consider an expression $A \vDash_{\mathbf{Cl}} B$ as an assertion about classical consequence relation according to a standard definition of a classical consequence relation.

Given a valuation $v\colon \mathtt{Var} \mapsto \mathscr{U}$ we define a corresponding classical valuation v^*:

$$v^*(p) = \begin{cases} t, & \text{if } v(p) \in \mathcal{D}, \\ f & \text{otherwise}, \end{cases}$$

[6] In the context of the current research a translation function Φ from the language of a binary consequence system S_1 to the language of a binary consequence system S_2 is an *embedding* when it holds that $A \vDash_{S_1} B \iff \Phi(A) \vDash_{S_2} \Phi(B)$. The are some other terms for similar kind of translations in the literature, see eg. [11, 7, 12].

where $p \in \texttt{Var}$. Now let v_1, v_2 and v_3 be extensions of classical-, $\mathbf{CNL^2_4}$- and $\mathbf{CNLL^2_4}$-valuations correspondingly (in the sequel we tacitly assume that a valuation v_1, v_2 or v_3 is an extended one when applied to formulas). It is not difficult to verify that the following lemma holds (in what follows 't.c.' stands for 'truth conditions', 'IH' for 'induction hypothesis').

Lemma 4.1. *For any formula $A \in \texttt{For}_{cl}$, any valuation v_2 (valuation v_3) there is a valuation v_1 such that $v_1(A) = t \iff v_2(\Phi(A)) \in \mathcal{D}$, ($v_1(A) = t \iff v_3(\Phi(A)) \in \mathcal{D}$).*

Proof: Simple reasoning by complexity of a formula A. Let us consider some cases, focusing on a valuation v_2 only.

Case $A = \neg B$.

$v_1(\neg B) = t \overset{\text{t.c.} \neg}{\iff} v_1(B) \neq t \overset{\text{IH}}{\iff} v_2(\Phi(B)) \notin \mathcal{D} \overset{\text{lem. 5.7}}{\iff} v_2(\sim\sim\Phi(B)) \in \mathcal{D}$.

Case $A = B \wedge C$.

$v_1(B \wedge C) = t \overset{\text{t.c.} \wedge}{\iff} v_1(B) = t$ and $v_1(C) = t \overset{\text{IH}}{\iff} v_2(\Phi(B)) \in \mathcal{D}$ and $v_2(\Phi(C)) \in \mathcal{D} \overset{\text{lem. 5.7}}{\iff} v_2(\Phi(B \wedge C)) \in \mathcal{D}$. □

We also need the converse of the previous lemma. Given that $v_1(p) = t$ for some $p \in \texttt{Var}$ we can choose a valuation v_2 (a valuation v_3) such that $v_2(p) \in \mathcal{D}$ ($v_3(p) \in \mathcal{D}$). Then it is easy to get the following lemma.

Lemma 4.2. *For any formula $A \in \texttt{For}_{cl}$, a classical valuation v_1, there exists a valuation v_2 (resp. a valuation v_3) such that*

$$v_1(A) = t \iff v_2(\Phi(A)) \in \mathcal{D} \quad (\textit{resp. } v_3(\Phi(A)) \in \mathcal{D}).$$

Lemma 4.3. *For all formulas $A, B \in \texttt{For}_{cl}$*

(1) $A \vDash_{\mathbf{Cl}} B \iff \Phi(A) \vDash_{\mathbf{CNL^2_4}} \Phi(B)$

(2) $A \vDash_{\mathbf{Cl}} B \iff \Phi(A) \vDash_{\mathbf{CNLL^2_4}} \Phi(B)$.

Proof: We consider $\mathbf{CNL^2_4}$ part. Let $A \models_{\mathbf{Cl}} B$, but $\Phi(A) \not\models_{\mathbf{CNL^2_4}} \Phi(B)$. Then there is a valuation v_2 such that $v_2(\Phi(A)) \in \mathcal{D}$, $v_2(\Phi(B)) \notin \mathcal{D}$. Applying lemma 4.1 we find a classical valuation v_1 such that $v_1(A) = t$, $v_1(B) \neq t$. The other direction is also clear. □

Corollary 4.4. Φ is an embedding of $\mathbf{Cl}$ into $\mathbf{CNL^2_4}$ ($\mathbf{CNLL^2_4}$).

What about the converse? Can we non-trivially translate our systems of cyclic negation to classical logic? To address this question let us define the following function Ψ, where i is a positive integer, A and B are formulas of the language $\mathcal{L}_{cnl}$:

$$\begin{aligned}
\Psi(p_i) &= p_{2i-1},\\
\Psi({\sim}p_i) &= p_{2i},\\
\Psi({\sim}{\sim}A) &= \neg\Psi(A),\\
\Psi(A \circ B) &= \Psi(A) \circ \Psi(B), \quad \circ \in \{\wedge, \vee\},\\
\Psi({\sim}(A \circ B)) &= \Psi({\sim}A) \circ \Psi({\sim}B), \quad \circ \in \{\wedge, \vee\}.
\end{aligned}$$

Similarly to the construction of a classical valuation v^* that has been used before, here we define (where i is a positive integer, $v\colon \texttt{Var} \mapsto \mathscr{U}$)

$$v^*(p_i) = \begin{cases} t, & \text{if } i \text{ is odd and } v(p_{\frac{i+1}{2}}) \in \mathcal{D},\\ t, & \text{if } i \text{ is even and } v(p_{\frac{i}{2}}) \in \{\mathbf{T}, \mathbf{FU}\},\\ f & \text{otherwise.} \end{cases}$$

We proceed with the following

Lemma 4.5. *For every* $\mathbf{CNL_4^2}$*-valuation* v_2 *and a formula* $A \in \texttt{For}$ *there exists a classical valuation* v_1 *such that*

$$v_1(\Psi(A)) = t \iff v_2(A) \in \mathcal{D}.$$

Proof: Let us consider firstly the case when $\Psi(A)$ is a propositional variable, say p_k. If k is odd index, then the statement follows from definition of v^*. If k is even, then suppose that $v_1(p_k) = v^*(p_k) = t$. Since preimage of p_k is ${\sim}p_{\frac{k}{2}}$ and $v_2(p_{\frac{k}{2}}) = v(p_{\frac{k}{2}}) \in \{\mathbf{T}, \mathbf{FU}\}$, $v_2({\sim}p_k) \in \mathcal{D}$. Other direction is evident.

Next let us consider some cases. Simple sub-cases are omitted.

Case $A = {\sim}{\sim}B$.

$v_1(\Psi({\sim}{\sim}B)) = t \overset{\text{df. }\Psi}{\iff} v_1(\neg(\Psi(B))) = t \overset{\text{t.c. }\neg}{\iff} v_1(\Psi(B)) \neq t \overset{\text{IH}}{\iff} v_2(B) \notin \mathcal{D} \overset{\text{t.c. }\sim}{\iff} v_2({\sim}{\sim}B) \in \mathcal{D}$.

Case $A = {\sim}(B \wedge C)$.

$v_1(\Psi({\sim}(B\wedge C))) = t \overset{\text{df. }\Psi}{\iff} v_1(\Psi({\sim}B)\wedge\Psi({\sim}C))) = t \overset{\text{t.c. }\wedge}{\iff} v_1(\Psi({\sim}B)) = t$ and $v_1(\Psi({\sim}C)) = t \overset{\text{IH}}{\iff} v_2({\sim}B) \in \mathcal{D}$ and $v_2({\sim}C) \in \mathcal{D} \overset{\text{t.c. }\sim}{\iff} v_2(B) \in \{\mathbf{T}, \mathbf{FU}\}$ and $v_2(C) \in \{\mathbf{T}, \mathbf{FU}\} \overset{\text{t.c. }\wedge}{\iff} v_2(B\wedge C) \in \{\mathbf{T}, \mathbf{FU}\} \overset{\text{t.c. }\sim}{\iff} v_2({\sim}(B\wedge C)) \in \mathcal{D}$.

CASE $A = {\sim}(B \vee C)$.

$v_1(\Psi({\sim}(B \vee C))) = t \overset{\text{df. }\Psi}{\Longleftrightarrow} v_1(\Psi({\sim}B) \vee \Psi({\sim}C))) = t \overset{\text{t.c. }\vee}{\Longleftrightarrow} v_1(\Psi({\sim}B)) = t$ or $v_1(\Psi({\sim}C)) = t \overset{\text{IH}}{\Longleftrightarrow} v_2({\sim}B) \in \mathcal{D}$ or $v_2({\sim}C) \in \mathcal{D} \overset{\text{t.c. }\sim}{\Longleftrightarrow} v_2(B) \in \{\mathbf{T}, \mathbf{FU}\}$ or $v_2(C) \in \{\mathbf{T}, \mathbf{FU}\} \overset{\text{t.c. }\vee}{\Longleftrightarrow} v_2(B \vee C) \in \{\mathbf{T}, \mathbf{FU}\} \overset{\text{t.c. }\sim}{\Longleftrightarrow} v_1({\sim}(B \vee C)) \in \mathcal{D}$. □

On the other hand, given a classical valuation v^* we can get a $\mathbf{CNL^2_4}$-valuation choosing an arbitrary mapping v such that $v(p_{\frac{i+1}{2}}) \in \mathcal{D}$ when $v^*(p_i) = t$ and $v(p_{\frac{i+1}{2}}) \notin \mathcal{D}$ when $v^*(\Psi(p_i)) = f$ for a an odd integer i, while $v(p_{\frac{i}{2}}) \in \{\mathbf{T}, \mathbf{FU}\}$ when $v^*(p_i) = t$ and $v(p_{\frac{i}{2}}) \notin \{\mathbf{T}, \mathbf{FU}\}$ when $v^*(p_i) = f$ for an even integer i. Thus we obtain an analogue of the previous lemma.

LEMMA 4.6. *For every classical valuation v_1 and a formula $A \in$ `For` there exists a $\mathbf{CNL^2_4}$-valuation v_2 such that*

$$v_1(\Psi(A)) = t \iff v_2(A) \in \mathcal{D}.$$

PROOF: Similar to the proof of the lemma 4.5 □

LEMMA 4.7. $A \models_{\mathbf{CNL^2_4}} B \iff \Psi(A) \models_{\mathbf{Cl}} \Psi(B)$.

PROOF: First assume that $\Psi(A) \models_{\mathbf{Cl}} \Psi(B)$, but $A \not\models_{\mathbf{CNL^2_4}} B$. Then there exists some extended $\mathbf{CNL^2_4}$-valuation v_2 such that $v_2(A) \in \mathcal{D}$ and $v_2(B) \notin \mathcal{D}$. According to lemma 4.5 there exists an extended classical valuation v_1 such that $v_1(\Psi(A)) = t$, but $v_1(\Psi(B)) = f$.

For the other direction suppose that $A \models_{\mathbf{CNL^2_4}} B$, but $\Psi(A) \not\models_{\mathbf{Cl}} \Psi(B)$. Then there exists a classical valuation v such that $v(\Psi(A)) = t$, $v(\Psi(B)) = f$. Using lemma 4.6 we conclude that $A \not\models_{\mathbf{CNL^2_4}} B$. □

COROLLARY 4.8. Ψ is an embedding of $\mathbf{CNL^2_4}$ into $\mathbf{Cl}$.

To obtain the same result for $\mathbf{CNLL^2_4}$ we need some modification of Ψ. But this time things appear to be far more complicated and, as it seems, there is no simple and elegant translation clauses for the negated $\wedge$ and $\vee$. Nevertheless, technically, it is still possible to define a required function. Let us denote by Ψ' a translation which differs from Ψ in what concerns the images of formulas of the form $\sim(B \wedge C)$ and $\sim(B \vee C)$ and agrees with it in other respects. Namely we put

$$\begin{aligned}\Psi'(\sim(B \wedge C)) = (\neg\Psi'(B) \wedge \Psi'(\sim B)) \vee (\neg\Psi'(C) \wedge \Psi'(\sim C)) \vee \\ \vee (\Psi'(\sim B) \wedge \Psi'(\sim C)), \\ \Psi'(\sim(B \vee C)) = (\Psi'(B) \wedge \Psi'(\sim B)) \vee (\Psi'(C) \wedge \Psi'(\sim C)) \vee \\ \vee (\Psi'(\sim B) \wedge \Psi'(\sim C)).\end{aligned}$$

For this translation we can prove analogues of lemmas 4.5 and 4.6. Let us denote as '*tr*' the right parts of the above equations when they are clear from the context.

Lemma 4.9. *For every extended* $\mathbf{CNLL_4^2}$*-valuation* v_3 *and a formula* $A \in$ `For` *there exists a classical valuation* v_1 *such that*

$$v_1(\Psi'(A)) = t \iff v_3(A) \in \mathcal{D}.$$

Proof: Let us check some crucial cases.

Case $A = \sim(B \wedge C)$.

First we have $v_1(\Psi'(\sim(B \wedge C))) = t \overset{\text{df. }\Psi'}{\Rightarrow} v_1(tr) = t$. Thus any disjunct of tr may be evaluated as t under v_1. Let us inspect all three sub-cases. We start with $v_1(\neg\Psi'(B) \wedge \Psi'(\sim B)) = t \overset{\text{t.c. }\wedge,\neg}{\Rightarrow} v_1(\Psi'(B)) = f$ and $v_1(\Psi'(\sim B)) = t \overset{\text{IH}}{\Rightarrow} v_3(B) \notin \mathcal{D}$ and $v_3(\sim B) \in \mathcal{D} \overset{\text{t.c. }\sim}{\Rightarrow} v_3(B) = \mathbf{FU} \overset{\text{t.c. }\wedge}{\Rightarrow} v_3(B \wedge C) = \mathbf{FU} \overset{\text{t.c. }\sim}{\Rightarrow} v_3(\sim(B \wedge C)) = \mathbf{T} \in \mathcal{D}$. The second disjunctive sub-case is similar.

Next consider the following implications: $v_1(\Psi'(\sim B) \wedge \Psi'(\sim C)) = t \overset{\text{t.c. }\wedge}{\Rightarrow} v_1(\Psi'(\sim B)) = t$ and $v_1(\Psi'(\sim C)) = t \overset{\text{IH}}{\Rightarrow} v_3(\sim B) \in \mathcal{D}$ and $v_3(\sim C) \in \mathcal{D} \overset{\text{t.c. }\sim}{\Rightarrow} v_3(B) \in \{\mathbf{T}, \mathbf{FU}\}$ and $v_3(C) \in \{\mathbf{T}, \mathbf{FU}\} \overset{\text{t.c. }\wedge}{\Rightarrow} v_3(B \wedge C) \in \{\mathbf{T}, \mathbf{FU}\} \overset{\text{t.c. }\sim}{\Rightarrow} v_3(\sim(B \wedge C)) \in \mathcal{D}$.

For the other direction $v_3(\sim(B \wedge C)) \in \mathcal{D} \overset{\text{t.c. }\sim}{\Rightarrow} v_3(B \wedge C) \in \{\mathbf{T}, \mathbf{FU}\} \overset{\text{t.c. }\wedge}{\Rightarrow}$ (a) $v_3(B) = v_3(C) = \mathbf{T}$ or (b) $v_3(B) = \mathbf{FU}$ or (c) $v_3(C) = \mathbf{FU}$.

Sub-case (a): $v_3(\sim B) = \mathbf{TU} \in \mathcal{D}$ and $v_3(\sim C) = \mathbf{TU} \in \mathcal{D} \overset{\text{IH}}{\Rightarrow} v_1(\Psi'(\sim B)) = t$ and $v_1(\Psi'(\sim C)) = t \overset{\text{t.c. }\wedge}{\Rightarrow} v_1(\Psi'(\sim B) \wedge \Psi'(\sim C)) = t \overset{\text{t.c. }\vee}{\Rightarrow} v_1(tr) = t \overset{\text{df. }\Psi'}{\Rightarrow} v_1(\Psi'(\sim(B \wedge C))) = t$.

Sub-case (b): $v_3(B) = \mathbf{FU} \notin \mathcal{D} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(\sim B) = \mathbf{T} \in \mathcal{D} \overset{\text{IH}}{\Rightarrow} v_1(\Psi'(B)) \neq t$ and $v_1(\Psi'(\sim B)) = t \overset{\text{t.c.}\wedge}{\Rightarrow} v_1(\neg\Psi'(B) \wedge \Psi'(\sim B)) = t \overset{\text{t.c.}\vee}{\Rightarrow} v_1(tr) = t \overset{\text{df.}\Psi'}{\Rightarrow} v_1(\Psi'(\sim(B \wedge C))) = t$. Sub-case (c) is similar.

Case $A = \sim(B \vee C)$.

$v_1(\Psi'(\sim(B \vee C))) = t \overset{\text{df.}\Psi'}{\Rightarrow} v_1(tr) = t$. Again, any disjunct of tr may have the value t under v_1. Consider the following sequence of implications: $v_1(\Psi'(B) \wedge \Psi'(\sim B)) = t \overset{\text{t.c.}\wedge}{\Rightarrow} v_1(\Psi'(B)) = t$ and $v_1(\Psi'(\sim B)) = t \overset{\text{IH}}{\Rightarrow} v_3(B) \in \mathcal{D}$ and $v_3(\sim B) \in \mathcal{D} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(B) = \mathbf{T} \overset{\text{t.c.}\vee}{\Rightarrow} v_3(B \vee C) = \mathbf{T} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(\sim(B \vee C)) = \mathbf{TU} \in \mathcal{D}$. The second disjunctive sub-case is similar, while the third one can be easily seen from the analogues sub-case for $\sim(B \wedge C)$.

For the other direction $v_3(\sim(B \vee C)) \in \mathcal{D} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(B \vee C) \in \{\mathbf{T}, \mathbf{FU}\} \overset{\text{t.c.}\vee}{\Rightarrow}$ (a) $v_3(B) = v_3(C) = \mathbf{FU}$ or (b) $v_3(B) = \mathbf{T}$ or (c) $v_3(C) = \mathbf{T}$.

Sub-case (a): $v_3(\sim B) = \mathbf{T} \in \mathcal{D}$ and $v_3(\sim C) = \mathbf{T} \in \mathcal{D} \overset{\text{IH}}{\Rightarrow} v_1(\Psi'(\sim B)) = t$ and $v_1(\Psi'(\sim C)) = t \overset{\text{t.c.}\wedge}{\Rightarrow} v_1(\Psi'(\sim B) \wedge \Psi'(\sim C)) = t \overset{\text{t.c.}\vee}{\Rightarrow} v_1(tr) = t \overset{\text{df.}\Psi'}{\Rightarrow} v_1(\Psi'(\sim(B \vee C))) = t$.

Sub-case (b): $v_3(B) = \mathbf{T} \in \mathcal{D} \overset{\text{t.c.}\sim}{\Rightarrow} v_3(\sim B) = \mathbf{TU} \in \mathcal{D} \overset{\text{IH}}{\Rightarrow} v_1(\Psi'(B)) = t$ and $v_1(\Psi'(\sim B)) = t \overset{\text{t.c.}\wedge}{\Rightarrow} v_1(\Psi'(B) \wedge \Psi'(\sim B)) = t \overset{\text{t.c.}\vee}{\Rightarrow} v_1(tr) = t \overset{\text{df.}\Psi'}{\Rightarrow} v_1(\Psi'(\sim(B \vee C))) = t$. Sub-case (c) is similar. □

Thus the following two lemmas are readily following.

Lemma 4.10. *For every classical valuation v_1 and a formula $A \in \texttt{For}$ there exists a $\mathbf{CNLL_4^2}$-valuation v_3 such that*

$$v_1(\Psi'(A)) = t \iff v_3(A) \in \mathcal{D}.$$

Lemma 4.11. $A \models_{\mathbf{CNLL_4^2}} B \iff \Psi'(A) \models_{\mathbf{Cl}} \Psi'(B)$.

Proof: Similar to the proof of lemma 4.7. □

Corollary 4.12. Ψ' is an embedding of $\mathbf{CNLL_4^2}$ into $\mathbf{Cl}$.

5. Soundness and completeness of $\mathbf{CNL_4^2}$ and $\mathbf{CNLL_4^2}$

5.1. Soundness

Lemma 5.1 (Local Soundness for $\mathbf{CNL_4^2}$). *All axiomatic schemata of $\mathbf{CNL_4^2}$ represent $\mathbf{CNL_4^2}$-valid sequents and the rules of inference preserve $\mathbf{CNL_4^2}$-validity.*

Proof: We need to check each item from the list of axiomatic schemata and inference rules. Let us show a couple of cases. Here, again, a valuation applied to formulas is just an extended valuation function.

Suppose that axiomatic schemata (a7) is invalid, i. e. there a $\mathbf{CNL_4^2}$-valuation v that $v(A \wedge {\sim}^2 A) \in \{\mathbf{T}, \mathbf{TU}\}$ and $v(B) \notin \{\mathbf{T}, \mathbf{TU}\}$ that is $v(B) \in \{\mathbf{F}, \mathbf{FU}\}$. It can be seen that this situation is impossible since $A \wedge {\sim}^2 A$ cannot take its value from the set $\{\mathbf{T}, \mathbf{TU}\}$ at all.

Suppose that the rule (r4) does not preserve validity. This means that there is a valuation v that $A \vDash_{\mathbf{CNL_4^2}} B$, but ${\sim}^2 B \nvDash_{\mathbf{CNL_4^2}} {\sim}^2 A$. From the latter it follows that t $v({\sim}^2 B) \in \{\mathbf{T}, \mathbf{TU}\}$ and $v({\sim}^2 A) \notin \{\mathbf{T}, \mathbf{TU}\}$ which means that $v({\sim}^2 A) \in \{\mathbf{F}, \mathbf{FU}\}$. It is easy to observe that definition of $\sim$ implies $v(A) \in \{\mathbf{T}, \mathbf{TU}\}$ and $v(B) \in \{\mathbf{F}, \mathbf{FU}\}$, but this contradicts to $A \vDash_{\mathbf{CNL_4^2}} B$. Therefore, (r4) preserves validity.

The other cases are similar. □

Theorem 5.2 (Soundness for $\mathbf{CNL_4^2}$). *For any formulas A and B of the language $\mathcal{L}_{cnl}$, the following holds:*

$$A \vdash B \text{ is } \mathbf{CNL_4^2}\text{-provable} \Rightarrow A \vDash_{\mathbf{CNL_4^2}} B.$$

Proof: By induction on the length of the proof, using Lemma 5.1. □

Lemma 5.3 (Local Soundness for $\mathbf{CNLL_4^2}$). *All axiomatic schemata of $\mathbf{CNLL_4^2}$ represent $\mathbf{CNLL_4^2}$-valid sequents and the rules of inference preserve $\mathbf{CNLL_4^2}$-validity.*

Proof: Analogously to Lemma 5.1, we show only an example with one axiomatic schemata, because the sets of inference rules of $\mathbf{CNL_4^2}$ and $\mathbf{CNLL_4^2}$ are identical.

Suppose that axiomatic schemata (c8) is invalid, that is there is a valuation v such that $v({\sim}(A \vee B) \wedge {\sim}(A \wedge B)) \in \{\mathbf{T}, \mathbf{TU}\}$ and $v({\sim}A \wedge {\sim}B) \notin \{\mathbf{T}, \mathbf{TU}\}$. The latter means that $v({\sim}A \wedge {\sim}B) \in \{\mathbf{F}, \mathbf{FU}\}$.

(a) Let $v({\sim}(A \vee B) \wedge {\sim}(A \wedge B)) = \mathbf{T}$. According to the definition of conjunction this means that $v({\sim}(A \vee B)) = \mathbf{T}$ and $v({\sim}(A \wedge B)) = \mathbf{T}$. This means that $v(A \vee B) = \mathbf{FU}$ and $v(A \wedge B) = \mathbf{FU}$. The first equation determines $v(A) = \mathbf{FU}$ and $v(B) = \mathbf{FU}$.

    Let $v({\sim}A \wedge {\sim}B) = \mathbf{F}$. This is possible when $v({\sim}A) = \mathbf{F}$ or $v({\sim}B) = \mathbf{F}$. That is $v(A) = \mathbf{TU}$ or $v(B) = \mathbf{TU}$. Each of these cases incompatible with the previous observation.

Let $v(\sim A \wedge \sim B) = \mathbf{FU}$. It takes place when $v(\sim A) = \mathbf{FU}$ or $v(\sim B) = \mathbf{FU}$ which implies $v(A) = \mathbf{F}$ or $v(B) = \mathbf{F}$, impossible again.

(b) Let $v(\sim(A \vee B) \wedge \sim(A \wedge B)) = \mathbf{TU}$. According to the definition of conjunction three cases are to consider, but two of them are identical. Suppose, $v(\sim(A \vee B)) = \mathbf{T}$ and $v(\sim(A \wedge B)) = \mathbf{TU}$. By truth conditions of $\sim$, $v(A \wedge B) = \mathbf{T}$. This means that $v(A) = \mathbf{T}$ and $v(B) = \mathbf{T}$. Inspecting already considered cases when $v(\sim A \wedge \sim B) \in \{\mathbf{F}, \mathbf{FU}\}$ we arrive at impossible valuations. The argument is analogous, when $v(\sim(A \vee B)) = \mathbf{TU}$ and $v(\sim(A \wedge B)) = \mathbf{TU}$. □

THEOREM 5.4 (Soundness for $\mathbf{CNLL^2_4}$). *For any formulas A and B of the language $\mathcal{L}_{cnl}$, the following holds:*

$$A \vdash B \textit{ is } \mathbf{CNLL^2_4}\textit{-provable} \Rightarrow A \models_{\mathbf{CNLL^2_4}} B.$$

PROOF: By induction on the length of the proof, using Lemma 5.3. □

5.2. Completeness

The idea of the completeness theorem proof is based on a technique elaborated by J. M. Dunn for the system of **FDE** (see [9]). This method essentially relies on the notion of a prime theory which is given in the following definition.

DEFINITION 5.5. A $\mathbf{CNL^2_4}$-($\mathbf{CNLL^2_4}$)-theory is the set of formulas α such that for all formulas A and B of the language $\mathcal{L}_{cnl}$,

(1) $A \wedge B \in \alpha$ whenever $A \in \alpha$ and $B \in \alpha$,

(2) $B \in \alpha$ whenever $A \in \alpha$ and $A \vdash B$ is $\mathbf{CNL^2_4}$-($\mathbf{CNLL^2_4}$)-provable.

A $\mathbf{CNL^2_4}$-($\mathbf{CNLL^2_4}$)-theory is prime if $A \vee B \in \alpha$ implies $A \in \alpha$ or $B \in \alpha$. We call a $\mathbf{CNL^2_4}$-($\mathbf{CNLL^2_4}$)-theory α *c-normal* when for each formula A it holds that $A \in \alpha$ if and only if $\sim^2 A \notin \alpha$.

As a first step toward completeness theorems for $\mathbf{CNL^2_4}$ and $\mathbf{CNLL^2_4}$ we prove the Extension Lemma. Note that we use this lemma uniformly for both completeness theorems. So we prove it for the case of $\mathbf{CNL^2_4}$, while proof for another system is the same.

LEMMA 5.6 (Extension Lemma). *For all formulas A and B of the language $\mathcal{L}_{cnl}$, if $A \vdash B$ is not $\mathbf{CNL_4^2}$-provable, then there is a c-normal prime theory α such that $A \in \alpha$, $B \notin \alpha$.*

PROOF: Suppose that for some formulas A and B, $A \vdash B$ is not $\mathbf{CNL_4^2}$-provable. Let us define $\alpha_0 = \{C \mid A \vdash_{\mathbf{CNL_4^2}} C\}$. α_0 is a theory as it is closed under $\vdash_{\mathbf{CNL_4^2}}$ and $\wedge$ (using the rule (r2)). Next we construct the sequence of theories taking some enumeration of the set `For` $(A_1, A_2, \ldots)$ and define

$$\alpha_{n+1} = \begin{cases} \alpha_n, \text{ if } \alpha_n \cup \{A_{n+1}\} \vdash_{\mathbf{CNL_4^2}} B, \\ \alpha_n \cup \{A_{n+1}\}, \text{ if } \alpha_n \cup \{A_{n+1}\} \nvdash_{\mathbf{CNL_4^2}} B. \end{cases}$$

Let α be the union of all α_n's. First we show that α is a prime theory such that $A \in \alpha$ and $B \notin \alpha$. $A \in \alpha$ by construction. Assume $B \in \alpha$, hence B was added to α_i on i-th stage of construction of the sequence, which is impossible. For the primeness suppose that α is not prime, i. e. $C \vee D \in \alpha$, but $C \notin \alpha$ and $D \notin \alpha$. This means that both extensions $\alpha \cup \{C\}$ and $\alpha \cup \{D\}$ contain B. Then there is a conjunctions of formulas form α, say E, such that $E \wedge C \vdash_{\mathbf{CNL_4^2}} B$ and $E \wedge D \vdash_{\mathbf{CNL_4^2}} B$. From this, using $(r3)$, we derive $(E \wedge C) \vee (E \wedge D) \vdash_{\mathbf{CNL_4^2}} B$. Then, using (a8) and $(r1)$, we have $E \wedge (C \vee D) \vdash_{\mathbf{CNL_4^2}} B$, so $B \in \alpha$.

Finally, α is also c-normal. Indeed, if for some k, $A_k \in \alpha$ and $\sim^2 A_k \in \alpha$, then there is an α_i which contains $A_k \wedge \sim^2 A_k$ as well as B, due to axiom schema $A \wedge \sim^2 A \vdash B$, contrary to the assumption. On the other hand, primeness of α and derivable schema $B \vdash_{\mathbf{CNL_4^2}} A \vee \sim^2 A$ guarantee that for each A_k, one of two formulas, A_k and $\sim^2 A_k$, belongs to α. □

5.3. Completeness for $\mathbf{CNL_4^2}$

Recall that $\mathcal{A}$ denotes the set $\{\mathbf{TU}, \mathbf{F}\}$. We can express our truth-values in terms of $\mathcal{A}$ and $\mathcal{D}$ sets via the following expressions:

$$v(A) = \mathbf{T} \iff v(A) \in \mathcal{D} \text{ and } v(A) \notin \mathcal{A},$$
$$v(A) = \mathbf{TU} \iff v(A) \in \mathcal{D} \text{ and } v(A) \in \mathcal{A},$$
$$v(A) = \mathbf{F} \iff v(A) \notin \mathcal{D} \text{ and } v(A) \in \mathcal{A},$$
$$v(A) = \mathbf{FU} \iff v(A) \notin \mathcal{D} \text{ and } v(A) \notin \mathcal{A}.$$

It is not difficult to verify the next lemma, having in mind the interpretations of propositional connectives.

LEMMA 5.7. *Let* $A, B \in \texttt{For}$, *and* v *be a* $\mathbf{CNL_4^2}$*-valuation. Then, the following statements hold:*

(1) $v({\sim}A) \in \mathcal{D} \iff v(A) \notin \mathcal{A}$,

(2) $v({\sim}A) \in \mathcal{A} \iff v(A) \in \mathcal{D}$,

(3) $v(A \wedge B) \in \mathcal{D} \iff v(A) \in \mathcal{D}$ *and* $v(B) \in \mathcal{D}$,

(4) $v(A \wedge B) \in \mathcal{A} \iff v(A) \in \mathcal{A}$ *or* $v(B) \in \mathcal{A}$,

(5) $v(A \vee B) \in \mathcal{D} \iff v(A) \in \mathcal{D}$ *or* $v(B) \in \mathcal{D}$,

(6) $v(A \vee B) \in \mathcal{A} \iff v(A) \in \mathcal{A}$ *and* $v(B) \in \mathcal{A}$.

Now we turn to the definition of a $\mathbf{CNL_4^2}$-canonical valuation.

DEFINITION 5.8. For each c-normal prime theory α and propositional variable p we define a $\mathbf{CNL_4^2}$-canonical valuation v^c as a mapping $\texttt{Var} \mapsto 4\mathcal{Q}$ satisfying the following expressions:

(1) $v^c(p) \in \mathcal{D} \iff p \in \alpha$;

(2) $v^c(p) \in \mathcal{A} \iff {\sim}^3 p \in \alpha$;

We define a unique extension of v^c to the set of all formulas in the usual way and denote this extension by v^c as well. We prove that extended valuation behaves as expected with respect to the c-normal prime theories.

LEMMA 5.9 (Canonical Valuation Lemma for $\mathbf{CNL_4^2}$). *For each c-normal prime theory* α, *formula* A *and extended canonical* $\mathbf{CNL_4^2}$*-valuation* v^c *the following statements hold:*

(1) $v^c(A) \in \mathcal{D} \iff A \in \alpha$,

(2) $v^c(A) \in \mathcal{A} \iff {\sim}^3 A \in \alpha$.

PROOF: By induction on the structure of a formula A. The base case when A is a propositional variable follows from the definition 5.8. Let us explore the cases for the complex formulas. The induction hypothesis ('IH' in the sequel) claims that lemma is true for their proper subformulas. We also use the two basic properties of theories, namely, their closure under conjunction and the relation $\vdash_{\mathbf{CNL_4^2}}$ throughout the proof.

CASE $A = {\sim}B$.
$v^c({\sim}B) \in \mathcal{D} \overset{\text{lem.5.7}}{\Longleftrightarrow} v^c(B) \notin \mathcal{A} \overset{\text{IH}}{\Longleftrightarrow} {\sim}^3 B \notin \alpha \overset{c\text{-norm.}}{\Longleftrightarrow} {\sim}B \in \alpha$.
$v^c({\sim}B) \in \mathcal{A} \overset{\text{lem.5.7}}{\Longleftrightarrow} v^c(B) \in \mathcal{D} \overset{\text{IH}}{\Longleftrightarrow} B \in \alpha \overset{\text{(a9),(a10)}}{\Longleftrightarrow} {\sim}^4 B \in \alpha$.

CASE $A = B \wedge C$.
$v^c(B \wedge C) \in \mathcal{D} \overset{\text{lem.5.7}}{\Longleftrightarrow} v^c(B) \in \mathcal{D}$ and $v^c(C) \in \mathcal{D} \overset{\text{IH}}{\Longleftrightarrow} B \in \alpha$ and $C \in \alpha \overset{\text{df.}\alpha\text{,(a1),(a2)}}{\Longleftrightarrow} B \wedge C \in \alpha$.
$v^c(B \wedge C) \in \mathcal{A} \overset{\text{lem.5.7}}{\Longleftrightarrow} v^c(B) \in \mathcal{A}$ or $v^c(C) \in \mathcal{A} \overset{\text{IH}}{\Longleftrightarrow} {\sim}^3 B \in \alpha$ or ${\sim}^3 C \in \alpha \overset{c\text{-norm.}}{\Longleftrightarrow} {\sim}B \notin \alpha$ or ${\sim}C \notin \alpha \overset{\text{(a1),(a2),(a5),(b1)}}{\Longleftrightarrow} {\sim}(B \wedge C) \notin \alpha \overset{c\text{-norm.}}{\Longleftrightarrow} {\sim}^3(B \wedge C) \in \alpha$.

CASE $A = B \vee C$.
$v^c(B \vee C) \in \mathcal{D} \overset{\text{lem.5.7}}{\Longleftrightarrow} v^c(B) \in \mathcal{D}$ or $v^c(C) \in \mathcal{D} \overset{\text{IH}}{\Longleftrightarrow} B \in \alpha$ or $C \in \alpha \overset{\text{(a3),(a4),prim.}}{\Longleftrightarrow} B \vee C \in \alpha$.
$v^c(B \vee C) \in \mathcal{A} \overset{\text{lem.5.7}}{\Longleftrightarrow} v^c(B) \in \mathcal{A}$ and $v^c(C) \in \mathcal{A} \overset{\text{IH}}{\Longleftrightarrow} {\sim}^3 B \in \alpha$ and ${\sim}^3 C \in \alpha \overset{c\text{-norm.}}{\Longleftrightarrow} {\sim}B \notin \alpha$ and ${\sim}C \notin \alpha \overset{\text{(a3),(a4),(a6),(b2),prim.}}{\Longleftrightarrow} {\sim}(B \vee C) \notin \alpha \overset{c\text{-norm.}}{\Longleftrightarrow} {\sim}^3(B \vee C) \in \alpha$. □

THEOREM 5.10 (Completeness for $\mathbf{CNL_4^2}$). *For any formulas A and B of the language $\mathcal{L}_{cnl}$, the following holds:*

$$A \vDash_{\mathbf{CNL_4^2}} B \Rightarrow A \vdash B \textit{ is } \mathbf{CNL_4^2}\textit{-provable.}$$

PROOF: Suppose $A \vdash B$ is not $\mathbf{CNL_4^2}$-provable. Then, by Lemma 5.6, there is prime theory α such that $A \in \alpha$ and $B \notin \alpha$. Then, by Lemma 5.9, we know that $v^c(A) \in \mathcal{D}$ but $v^c(B) \notin \mathcal{D}$, so $A \nvDash_{\mathbf{CNL_4^2}} B$. □

5.4. Comleteness for $\mathbf{CNLL_4^2}$

Let $\mathcal{B}$ denote the set $\{\mathbf{T}, \mathbf{FU}\}$. The next lemma is rather straightforward consequence of the semantic definitions for the propositional connectives.

LEMMA 5.11. *For any $A, B \in$ `For`, a $\mathbf{CNLL_4^2}$-valuation v the following statements hold:*

(1) $v({\sim}A) \in \mathcal{D} \iff v(A) \in \mathcal{B}$,

(2) $v({\sim}A) \in \mathcal{B} \iff v(A) \notin \mathcal{D}$,

(3) $v(A \wedge B) \in \mathcal{D} \iff v(A) \in \mathcal{D}$ *and* $v(B) \in \mathcal{D}$,

(4) $v(A \wedge B) \in \mathcal{B} \iff v(A), v(B) \in \mathcal{D} \cap \mathcal{B}$ *or* $v(A) \in \mathcal{B} \setminus \mathcal{D}$ *or* $v(B) \in \mathcal{B} \setminus \mathcal{D}$,

(5) $v(A \vee B) \in \mathcal{D} \iff v(A) \in \mathcal{D}$ *or* $v(B) \in \mathcal{D}$,

(6) $v(A \vee B) \in \mathcal{B} \iff v(A), v(B) \in \mathcal{B} \setminus \mathcal{D}$ *or* $v(A) \in \mathcal{D} \cap \mathcal{B}$ *or* $v(B) \in \mathcal{D} \cap \mathcal{B}$.

Definition 5.12. For each c-normal prime theory α and propositional variable p we define a $\mathbf{CNLL_4^2}$-canonical valuation v^c as a mapping $\mathtt{Var} \mapsto 4\mathcal{LQ}$ satisfying the following expressions:

(1) $v^c(p) \in \mathcal{D} \iff p \in \alpha$;

(2) $v^c(p) \in \mathcal{B} \iff {\sim}p \in \alpha$;

Again, we need to extend a canonical valuation to the whole set `For` and prove the canonical valuation lemma.

Lemma 5.13 (Canonical Valuation Lemma for $\mathbf{CNLL_4^2}$). *For each c-normal prime theory α, formula A and extended canonical $\mathbf{CNLL_4^2}$-valuation v^c the following statements hold:*

(1) $v^c(A) \in \mathcal{D} \iff A \in \alpha$,

(2) $v^c(A) \in \mathcal{B} \iff {\sim}A \in \alpha$.

Proof: By induction on the structure of a formula. Propositional variables case immediately follows from the definition of v^c.

Case $A = {\sim}B$.

$v^c({\sim}B) \in \mathcal{D} \overset{\text{lem. 5.11}}{\iff} v^c(B) \in \mathcal{B} \overset{\text{IH}}{\iff} {\sim}B \in \alpha$.

$v^c({\sim}B) \in \mathcal{B} \overset{\text{lem. 5.11}}{\iff} v^c(B) \notin \mathcal{D} \overset{\text{IH}}{\iff} B \notin \alpha \overset{c\text{-norm.}}{\iff} {\sim}^2 B \in \alpha$.

Case $A = B \wedge C$.

$v^c(B \wedge C) \in \mathcal{D} \overset{\text{lem. 5.11}}{\iff} v^c(B) \in \mathcal{D}$ and $v^c(C) \in \mathcal{D} \overset{\text{IH}}{\iff} B \in \alpha$ and $C \in \alpha \overset{\text{df.}\alpha,\text{(a1)},\text{(a2)}}{\iff} B \wedge C \in \alpha$.

($\Rightarrow$) Let $v^c(B \wedge C) \in \mathcal{B}$. By Lemma 5.11 we have to explore three sub-cases. (i) From $[v^c(B) \in \mathcal{B}$ and $v^c(C) \in \mathcal{B}]$ and IH we get ${\sim}B \in \alpha$ and ${\sim}C \in \alpha$, thus by the $\wedge$-closure of α and the axiom scheme (a5), ${\sim}(B \wedge C) \in \alpha$. (ii) If $[v^c(B) \notin \mathcal{D}$ and $v^c(B) \in \mathcal{B}]$ then IH gives $B \notin \alpha$ and ${\sim}B \in \alpha$. By c-normality of α, ${\sim}^2 B \in \alpha$. Thus, by the axiom schema (c3), ${\sim}(B \wedge C) \in \alpha$. (iii) If $[v(C) \notin \mathcal{D}$ and $v(C) \in \mathcal{B}]$ we similarly get ${\sim}C \wedge {\sim}^2 C \in \alpha$, so ${\sim}(C \wedge B) \in \alpha$ and, finally, by (c7), ${\sim}(B \wedge C) \in \alpha$.

($\Leftarrow$) Suppose $\sim(B \wedge C) \in \alpha$, then, by (c2), $\sim B \vee \sim C \in \alpha$, so, by primeness of α, $\sim B \in \alpha$ or $\sim C \in \alpha$. Let us consider the case $\sim B \in \alpha$. According to IH, $v^c(B) \in \mathcal{B}$, but this is not enough to assert $v^c(B \wedge C) \in \mathcal{B}$. So, we should examine the position of B relative to the theory α. Suppose $B \in \alpha$. By the $\wedge$-closure of α, $B \wedge \sim B \in \alpha$. Using the axiom schema (c4) we get $\sim(B \vee C) \in \alpha$ which, along with (c8), and $\wedge$-closure of α again, implies $\sim B \wedge \sim C \in \alpha$, hence $\sim C \in \alpha$. By IH, $v^c(C) \in \mathcal{B}$, so $v^c(B \wedge C) \in \mathcal{B}$. Next assume $B \notin \alpha$. Applying IH we then have $v^c(B) \notin \mathcal{D}$. This means that $v^c(B) = \mathbf{FU}$, so $v^c(B \wedge C) \in \mathcal{B}$. Similarly for $\sim C \in \alpha$.

Case $A = B \vee C$.

$v^c(B \vee C) \in \mathcal{D} \overset{\text{lem. 5.11}}{\Longleftrightarrow} v^c(B) \in \mathcal{D}$ or $v^c(C) \in \mathcal{D} \overset{\text{IH}}{\Longleftrightarrow} B \in \alpha$ or $C \in \alpha \overset{\text{(a3),(a4),prim.}}{\Longleftrightarrow} B \vee C \in \alpha$.

($\Rightarrow$) Assume $v^c(B \vee C) \in \mathcal{B}$. Then, according to Lemma 5.11, we have two disjunctive subcases. First assume $[v^c(B) \in \mathcal{B}$ and $v^c(C) \in \mathcal{B}]$. It is enough to get $\sim B \in \alpha$ and $\sim C \in \alpha$ by IH and then $\sim(B \vee C) \in \alpha$ using (c1). The proof for second subcase is accomplished by the same reasoning.

($\Leftarrow$) Suppose $\sim(B \vee C) \in \alpha$. By the axiom (a6) and primeness of α we then obtain $\sim B \in \alpha$ or $\sim C \in \alpha$. Let us consider the first of the disjunctive sub-cases. From IH it follows that $v^c(B) \in \mathcal{B}$. But to get the required assertion $v^c(B \vee C) \in \mathcal{B}$ we need more information. Applying (c6) and then (c5) to $\sim(B \vee C) \in \alpha$ we get $\sim C \vee B \in \alpha$. Primeness of α and IH give $v^c(C) \in \mathcal{B}$ or $v^c(B) \in \mathcal{D}$. In both of these cases, taking into account $v^c(B) \in \mathcal{B}$, we end with $v^c(B \vee C) \in \mathcal{B}$. Analogues reasoning provides the proof in case when $\sim C \in \alpha$. □

Theorem 5.14 (Completeness for $\mathbf{CNLL_4^2}$). *For any formulas A and B of the language $\mathcal{L}_{cnl}$,*

$$A \models_{\mathbf{CNLL_4^2}} B \Rightarrow A \vdash B \textit{ is } \mathbf{CNLL_4^2}\textit{-provable.}$$

Proof: The same as in the previous theorem for $\mathbf{CNL_4^2}$. □

6. Conclusion

Although we have studied probably the most natural logics of paired cyclic negations, the whole picture is still waiting to be explored. Even the framework of the four-valued semantics gives some possible directions for the further investigations. Specifically, one can choose other sets of the designated truth values or combine the different collections of designated and anti-designated truth values. On the other hand, alternative definitions of the consequence relation are also possible. To obtain the more abstract results, paired cyclic negations could be put into more general lattice structures, even not necessary finitely based. Having in mind ability to simulate the other negation-like operations, the potential relationships between logical systems appear to be of the main interest.

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