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CATEGORICAL DUALITIES FOR SOME TWO CATEGORIES OF LATTICES: AN EXTENDED ABSTRACT

Abstract

The categorical dualities presented are: (first) for the category of bi-algebraic lattices that belong to the variety generated by the smallest non-modular lattice with complete (0,1)-lattice homomorphisms as morphisms, and (second) for the category of non-trivial (0,1)-lattices belonging to the same variety with (0,1)-lattice homomorphisms as morphisms. Although the two categories coincide on their finite objects, the presented dualities essentially differ mostly but not only by the fact that the duality for the second category uses topology. Using the presented dualities and some known in the literature results we prove that the Q-lattice of any non-trivial variety of (0,1)-lattices is either a 2-element chain or is uncountable and non-distributive.

Keywords: Categorical duality, bi-algebraic lattice, bounded lattice, quasivariety lattice.

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1. Definitions and two key lemmas

Obtaining categorical duality results for certain categories of structures has a long history. The classical examples are the Stone and Priestley dualities

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for bounded distributive lattices and their many extensions for categories of algebras associated with non-classical logics the algebraic parts of which contain distributive lattices. In this note, we present two results of this nature. Each of them goes one step beyond distributivity. The variety of bounded lattices generated by the smallest non-modular lattice is one of the two minimal varieties that extended the variety of bounded distributive latices.

A bi-algebraic lattice is a non-trivial lattice that is algebraic and the lattice dual (by reversing the lattice order) is also algebraic. A (0,1)-lattice is a lattice in which 0 and 1 are the smallest and greatest elements in the lattice and they are included as constants to the signature of the lattice. Lattices of this type are called bounded lattices.

A Q-lattice is the lattice whose elements are the quasivarieties contained in a quasivariety. The lattice order of a Q-lattice is the inclusion. A quasivariety is a class of structures that is closed under the operators **S** of forming isomorphic substructures, Cartesian products **P**, and ultraproducts. A variety is a quasivariety that additionally is closed under the operator of forming homomorphic images.

The lattices N_5 and M_3 each of which has 5 elements are the smallest non-modular and modular but non-distributive lattices, respectively. They are regarded as $(0,1)$-lattices. It is known that the variety of bounded lattices generated by N_5 coincides with $\mathbf{SP}(N_5)$.

For a partially ordered set $\langle X, \leq\rangle$ and subsets Y, Z of X, we write $Y \ll Z$ to mean that for every $y \in Y$ there exists $z \in Z$ such that $y \leq z$.

For a lattice L, an element $a \in L$, and a finite subset X of L with a being below the lattice join in L of the elements of X, it is said that X is a *join cover* of a. If a is not below any element of X, it is said that X is a *non-trivial* join cover of X. A non-trivial join cover X of a in L is said to be *minimal* if, for every non-trivial join cover Y of a in L with $Y \ll X$, it follows that $X \subseteq Y$.

For a fuller account of concepts used in our note we refer to [9] and [11].

The four equations displayed below are valid in N_5 and so they are valid in every lattice belonging to $\mathbf{SP}(N_5)$. They contain the key information for what we need for the functors establishing the presented dualities to be well defined on the objects of the considered categories. What we need is stated in Lemmas 1.1 and 1.2.

The lattice equation D_2 is a particular case of the family of lattice equations D_n, $n \geqslant 2$, which was introduced in [12]. Lattices which satisfy D_n

are called *n-distributive*. In the presence of D_2 the equation C is equivalent to the equation τ'_{21}. The equation τ'_{21} belongs to the family of lattice equations τ'_{nk} constructed in [14].

$$\mathrm{C}\colon\ x \wedge (y_0 \vee y_1) \wedge (z_0 \vee z_1) = \bigvee_{i<2} \big[x \wedge y_i \wedge (z_0 \vee z_1)\big] \vee$$
$$\vee \bigvee_{i<2} \big[x \wedge z_i \wedge (y_0 \vee y_1)\big] \ \vee\ \vee \bigvee_{i<2} \big[x \wedge \big((y_0 \wedge z_i) \vee (y_1 \wedge z_{1-i})\big)\big];$$

$$\mathrm{D}_2\colon\ x \wedge (y_0 \vee y_1 \vee y_2) = \bigvee_{i \leqslant 2} \big[x \wedge \bigvee_{j \neq i} y_j\big];$$

$$\mathrm{N}_5^0\colon\ x \wedge (y_0 \vee y_1) = \bigvee_{i<2} \big[x \wedge \big((y_i \wedge x) \vee y_{1-i}\big)\big];$$

$$\mathrm{N}_5^1\colon\ x \wedge \big[\big(y_0 \wedge (z_0 \vee z_1)\big) \vee y_1\big] = \big[x \wedge y_0 \wedge (z_0 \vee z_1)\big] \vee \big[x \wedge y_1\big] \vee$$
$$\vee \bigvee_{i<2} \big[x \wedge \big((y_0 \wedge z_i) \vee y_1\big)\big].$$

LEMMA 1.1. *For a dually algebraic lattice L, the following conditions are equivalent.*

i) $L \in \mathbf{SP}(N_5)$.

ii) *For every join-irreducible element x of L that is not join-prime, there is a unique minimal non-trivial join cover $\{a, b\}$ of x such that both a and b are join-irreducible and join-prime and, moreover, they satisfy either $a < x$ and $\{x, b\}$ is an antichain or $b < x$ and $\{x, a\}$ is an antichain.*

PROOF (SKETCH): i) implies ii): The equations C and D_2 or, equivalently, τ'_{21} and D_2, by Theorems 3.2 and 3.4 of [14], together imply that every join-irreducible x of L has a unique minimal non-trivial join cover $\{a, b\}$. By minimality of $\{a, b\}$, a and b are join-irreducible. The equations N_5^0 and N_5^1 justify that the unique pair has the remaining properties as stated in ii).

ii) implies i): This implication is an easy consequence of the main result of [3]. It can also be proved without the result of [3] but with some effort. □

In every bi-algebraic lattice, every element is completely join-irreducible or is the lattice join of all completely join-irreducible elements that are below. Moreover, completely join-irreducible elements are compact. Lemma 1.2 stated below follows from Lemma 1.1.

LEMMA 1.2. *For a bi-algebraic lattice L, the following conditions are equivalent.*

i) $L \in \mathbf{SP}(N_5)$.

ii) *For every completely join-irreducible element x of L that is not join-prime, there is a unique minimal non-trivial join cover $\{a, b\}$ of x such that both a and b are completely join-irreducible and join-prime and, moreover, they satisfy either $a < x$ and $\{x, b\}$ is an antichain or $b < x$ and $\{x, a\}$ is an antichain.*

Lemma 1.2 is the key lemma in the construction of the functor $\mathsf{N}\colon \mathbb{B}_5 \to \mathbb{N}_5$ on the objects of $\mathbb{B}_5$ and, consequently, the functor $\mathsf{B}\colon \mathbb{N}_5 \to \mathbb{B}_5$ on the objects of $\mathbb{N}_5$ but after having (discovering) the precise definition of the category $\mathbb{N}_5$. Lemma 1.2 says how to define the function $f\colon Y \to X^2$ which is the most important ingredient in the definition of N_5-space (an object of $\mathbb{N}_5$) that is assigned to L (an object of $\mathbb{B}_5$).

Lemma 1.1 is the key lemma in the construction of the functor $\mathsf{T}\colon \mathbb{L}_5 \to \mathbb{T}_5$ on the objects of $\mathbb{L}_5$ and, consequently, the functor $\mathsf{L}\colon \mathbb{T}_5 \to \mathbb{L}_5$ on the objects of $\mathbb{T}_5$ and again after having (discovering) the precise definition of the category $\mathbb{T}_5$. Lemma 1.1 says how to define the function $f\colon Y(L) \to X(L)^2$ on the spectral N_5-space (an object of $\mathbb{T}_5$) assigned to L (an object of $\mathbb{L}_5$). In defining f, we use the known facts which say that any lattice L embeds into the lattice $F(L)$ of filters on L, $F(L)$ is dually algebraic, and that L and $F(L)$ satisfy the same lattice equations. A detailed description of the correctness of the presented dualities depend on the proof type context.

2. Categories $\mathbb{N}_5$ and $\mathbb{B}_5$

DEFINITION 2.1. A structure $\mathbb{S} = \langle X, Y, \leq, f\rangle$ is an N_5-*space*, if

(s1) $X \cup Y \neq \varnothing$ and $X \cap Y = \varnothing$; moreover, if $Y \neq \varnothing$, then $X \neq \varnothing$;

(s2) $\leq$ is a partial order on $X \cup Y$;

(s3) $f\colon Y \to X^2$ is a function and for all $y \in Y$ with $f(y) = (a, b)$, the following conditions hold:

(a) $a \leq y$ and $\{a, b\}$, $\{y, b\}$ are antichains;

(b) if $a, b \leq z$ for some $z \in X \cup Y$ then $y \leq z$;

(c) if $z \leq y$ for some $z \in X \cup Y$ then either $z \leq a$ or $z \leq b$, or $z \in Y$ and $\{u, v\} \ll \{a, b\}$ where $f(z) = (u, v)$.

Definition 2.2. Let $\mathbb{S} = \langle X, Y, \leq, f\rangle$ and $\mathbb{S}' = \langle X', Y', \leq', f'\rangle$ be N_5-spaces. A mapping $\varphi\colon \mathbb{S} \to \mathbb{S}'$ is an N_5-*morphism*, if the following conditions hold:

(m1) φ maps $X \cup Y$ into $X' \cup Y' \cup 2X'$, where $2X'$ denotes the collection of 2-element antichains in $\langle X', \leq'\rangle$

(m2) if $u, v \in X \cup Y$ are such that $\varphi(u), \varphi(v) \in X' \cup Y'$ and $u \leq v$ then $\varphi(u) \leq \varphi(v)$;

(m3) for all $x \in X$, $\varphi(x) \in X'$;

(m4) for all $y \in Y$ with $f(y) = (a, b)$, the following holds:

(a) if $\varphi(y) \in X'$, then either $\varphi(y) = \varphi(a)$ or $\varphi(y) \leq \varphi(b)$;

(b) if $\varphi(y) \in Y'$, then $f'\big(\varphi(y)\big) = \big(\varphi(a), \varphi(b)\big)$;

(c) if $\varphi(y) \in 2X'$, then $\varphi(y) = \big\{\varphi(a), \varphi(b)\big\}$ and, for every $z \in X \cup Y$ with $y \leq z$, one has $\varphi(a), \varphi(b) \leq' \varphi(z)$ if $\varphi(z) \in X' \cup Y'$, and $\{\varphi(a), \varphi(b)\} \ll \varphi(z)$ if $\varphi(z) \in 2X'$.

Proposition 2.3. Let $\varphi_0\colon \mathbb{S}_0 \to \mathbb{S}_1$, $\varphi_1\colon \mathbb{S}_1 \to \mathbb{S}_2$ be N_5-morphisms. The composition $\varphi_0 \circ \varphi_1\colon \mathbb{S}_0 \to \mathbb{S}_2$ of φ_0 and φ_1 in $\mathbb{N}_5$ is as follows, where $z \in X_0 \cup Y_0$.

(c1) If $\varphi_0(z) \in X_1 \cup Y_1$, then $\varphi_0 \circ \varphi_1(z) = \varphi_1\varphi_0(z)$.

(c2) If $\varphi_0(z) \in 2X_1$, then

$$\varphi_0 \circ \varphi_1(z) = \begin{cases} \varphi_1\varphi_0(u), & \text{if } \varphi_1\varphi_0(v) \leq \varphi_1\varphi_0(u);\\ \varphi_1\varphi_0(v), & \text{if } \varphi_1\varphi_0(u) \leq \varphi_1\varphi_0(v);\\ \{\varphi_1\varphi_0(u), \varphi_1\varphi_0(v)\}, & \text{if } \{\varphi_1\varphi_0(u), \varphi_1\varphi_0(v)\} \in 2X_2, \end{cases}$$

where $f(z) = (u, v)$ in $\mathbb{S}_0$.

The two categories $\mathbb{N}_5$ and $\mathbb{B}_5$ are as follows. Objects in $\mathbb{N}_5$ are N_5-spaces; morphisms are N_5-morphisms. Objects in $\mathbb{B}_5$ are bi-algebraic lattices belonging to the variety $\mathbf{SP}(N_5)$; morphisms are complete $(0, 1)$-lattice homomorphisms. In this section, we construct two contravariant functors $\mathsf{B}\colon \mathbb{N}_5 \to \mathbb{B}_5$ and $\mathsf{N}\colon \mathbb{B}_5 \to \mathbb{N}_5$ which establish duality between $\mathbb{N}_5$ and $\mathbb{B}_5$.

Definition 2.4. Let $\mathbb{S} = \langle X, Y, \leq, f\rangle$ be an N_5-space. A subset $I \subseteq X \cup Y$ is an *ideal of N_5-space* $\mathbb{S}$ if I is a lower cone with respect to $\leq$ and has the following property:

$$\text{if } f(y) = (a, b) \text{ in } \mathbb{S} \text{ and } a, b \in I \text{ then } y \in I.$$

The set of all ideals of $\mathbb{S}$ forms a complete $(0, 1)$-lattice with the lattice operations given by:

$$\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i;$$

$$\bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i \cup \{y \in Y \mid y = f(a, b) \text{ and } a, b \in \bigcup_{i\in I} X \cap A_i\}.$$

The functor $\mathsf{B}\colon \mathbb{N}_5 \to \mathbb{B}_5$ is defined as follows, where $\mathbb{S}$ and $\mathbb{S}'$ are N_5-spaces and $\varphi\colon \mathbb{S} \to \mathbb{S}'$ is an N_5-morphism:

$\mathsf{B}(\mathbb{S})$ is the complete $(0, 1)$-lattice defined above;

$\mathsf{B}(\varphi)\colon \mathsf{B}(\mathbb{S}') \to \mathsf{B}(\mathbb{S})$ is defined by $\mathsf{B}(\varphi)(Z') = \varphi^{-1}(Z')$.

Proposition 2.5. The following statements hold.

(1) $\mathsf{B}(\mathbb{S})$ is a bi-algebraic lattice that belongs to $\mathbf{SP}(N_5)$.

(2) $\mathsf{B}(\varphi)\colon \mathsf{B}(\mathbb{S}') \to \mathsf{B}(\mathbb{S})$ is a complete $(0, 1)$-lattice homomorphism.

Corollary 2.6. $\mathsf{B}\colon \mathbb{N}_5 \to \mathbb{B}_5$ is a contravariant functor.

For a lattice $L \in \mathbb{B}_5$, let

$$\mathsf{N}(L) = \langle X, Y, \leq, f\rangle,$$

where X is the set of all completely join-irreducible elements of L which are join-prime, Y is the set of all completely join-irreducible elements of L which are not join-prime, $\leq$ is the lattice order in L, and $f\colon Y \to X^2$ is a function that is defined as follows: $f(y) = (a, b)$, where $\{a, b\}$ is the unique pair of elements of X which, by Lemma 1.2, exists for y and, by choice, $a < y$.

For $L, L' \in \mathbb{B}_5$ and a complete lattice $(0,1)$-lattice homomorphism $g\colon L \to L'$, consider the map:

$$\beta_g\colon L' \to L, \quad \beta_g\colon a \mapsto \bigwedge\{b \in L' \mid g(b) = a\}.$$

We note that $g(\beta_g(a)) = a$ for all a of L'. We also note that if a is completely join-irreducible in L', then so is $\beta_g(a)$ but in L.

For a morphism $g\colon L \to L'$ in $\mathbb{B}_5$, we define $\mathsf{N}(g)\colon \mathsf{N}(L') \to \mathsf{N}(L)$ as follows:

$$\mathsf{N}(g)(y) = \begin{cases} \beta_g(y) & \text{if } \beta_g(y) \in X \cup Y; \\ \{\beta_g(a), \beta_g(b)\} & \text{if } \beta_g(y) \in 2X \text{ and } f(y) = (a,b) \end{cases}$$

$\mathsf{N}(L)$ and $\mathsf{N}(g)$ above define the second contravariant functor $\mathsf{N}\colon \mathbb{B}_5 \to \mathbb{N}_5$ justification of which follows from the proposition below.

Proposition 2.7. The following statements hold.

(1) $\mathsf{N}(L) \in \mathbb{N}_5$.

(2) $\mathsf{N}(g)\colon \mathsf{N}(L) \to \mathsf{N}(M)$ is an N_5-morphism.

Corollary 2.8. $\mathsf{N}\colon \mathbb{B}_5 \to \mathbb{N}_5$ is a contravariant functor.

Let $1_{\mathbb{N}_5}$ and $1_{\mathbb{B}_5}$ denote the identity functors within the categories $\mathbb{N}_5$ and $\mathbb{B}_5$, respectively.

Proposition 2.9. The pair NB and $1_{\mathbb{N}_5}$ as well as the pair BN and $1_{\mathbb{B}_5}$ are isomorphic functors.

Corollaries 2.6, 2.8 and Proposition 2.9 justify the following theorem.

Theorem 2.10. *The categories* $\mathbb{N}_5$ *and* $\mathbb{B}_5$ *are dually equivalent.*

The following corollary of Theorem 2.10 and the properties of the functors used show an advantage of dualities over algebraic approach if one wants to establish a technically demanding in proof result.

Corollary 2.11. The following statements hold.

(1) N_5-morphisms in $\mathbb{N}_5$ which are onto correspond by duality to one-to-one homomorphisms in $\mathbb{B}_5$ and *vice versa*.

(2) N_5-morphisms in $\mathbb{N}_5$ which are one-to-one correspond by duality to onto homomorphisms in $\mathbb{B}_5$ and *vice versa*.

(3) Disjoint unions of spaces (coproducts in $\mathbb{N}_5$) correspond by duality to Cartesian products in $\mathbb{B}_5$ and *vice versa*.

Let $(\mathbb{N}_5)_{fin}$ and $(\mathbb{B}_5)_{fin}$ denote the full subcategories in $\mathbb{N}_5$ and $\mathbb{B}_5$, respectively, whose objects are finite. From our construction of functors B and N and Theorem 2.10, we obtain

COROLLARY 2.12. The categories $(\mathbb{N}_5)_{fin}$ and $(\mathbb{B}_5)_{fin}$ are dually equivalent.

3. Categories $\mathbb{T}_5$ and $\mathbb{L}_5$

The two categories in this section $\mathbb{T}_5$ and $\mathbb{L}_5$ are as follows. Objects in $\mathbb{T}_5$ are spectral N_5-spaces; morphisms are spectral N_5-morphisms; see Definitions 3.1 and 3.4 provided below. Objects in $\mathbb{L}_5$ are bounded lattices belonging to the variety $\mathbf{SP}(N_5)$; morphisms are $(0,1)$-lattice homomorphisms. In this section, we construct two contravariant functors $\mathsf{L}\colon \mathbb{T}_5 \to \mathbb{L}_5$ and $\mathsf{T}\colon \mathbb{L}_5 \to \mathbb{T}_5$ which establish duality between $\mathbb{T}_5$ and $\mathbb{L}_5$.

We will consider pairs $(\mathbb{S}, \mathcal{T})$ such that $\mathbb{S} = \langle X, Y, \leq, f\rangle$ is a N_5-space and $\mathcal{T}$ is a topology on $X \cup Y$.

A subset A of $X \cup Y$ is said to be N_5-*compact* in $(\mathbb{S}, \mathcal{T})$ if the following conditions are satisfied:

(i) $A \cap X$ is compact in $(X, \{X \cap Z \mid Z \in \mathcal{T}\})$;

(ii) for every family $\{A_i \mid i \in I\}$ of open sets in $(X \cup Y, \mathcal{T})$, from $A \subseteq \bigcup_{i\in I} A_i$ it follows that $A \subseteq \bigcup_{i\in J} A_i \cup \{y \in Y \mid f(y) = (a, b) \text{ and } a, b \in \bigcup_{i\in J} X \cap A_i\}$ for some finite subset J of I.

We say that a subset A of $X \cup Y$ is f-*closed* in $\mathbb{S}$ if it is an ideal in $\mathbb{S}$; see Definition 2.4.

On the elements of every topological T_0-space with topology $\mathcal{T}$, there is a partial order $\leq_{\mathcal{T}}$ defined as follows: $x \leq_{\mathcal{T}} y$ iff every open set of $\mathcal{T}$ containing x contains y.

DEFINITION 3.1. A pair $(\mathbb{S}, \mathcal{T})$ is said to be a *spectral* N_5-*space* if the following conditions are fulfilled:

(1) $\mathbb{S}$ is a N_5-space, $\mathcal{T}$ is a T_0 topology on $X \cup Y$ the restriction to X of which makes X to be a spectral space, and $X \cup Y$ is N_5-compact in $(\mathbb{S}, \mathcal{T})$.

(2) $\leq = \leq_{\mathcal{T}}^{-1}$.

(3) The collection of all sets that are f-closed in $\mathbb{S}$, open in $(X \cup Y, \mathcal{F})$, and N_5-compact in $(\mathbb{S}, \mathcal{F})$ forms a basis for $(X \cup Y, \mathcal{T})$ that is closed under finite set intersections.

(4) For all sets A and B that are f-closed in $\mathbb{S}$, open in $(X \cup Y, \mathcal{F})$, and N_5-compact in $(\mathbb{S}, \mathcal{F})$, $A \cup B \cup \{y \in Y \mid f(y) = (a, b) \text{ and } a, b \in (A \cup B) \cap X\}$ is open in $(X \cup Y, \mathcal{T})$;

(5) $(\mathbb{S}, \mathcal{T})$ does not have a proper uN_5-extension; see Definition 3.10 provided below.

Remark 3.2. One can show that the set in the conclusion of (4) in Definition 3.1 is N_5-compact in $(\mathbb{S}, \mathcal{T})$ and, obviously, is f-closed in $\mathbb{S}$.

Remark 3.3. The set of all subsets of $X \cup Y$ that are f-closed in $\mathbb{S}$, open in $(X \cup Y, \mathcal{T})$, and N_5-compact in $(\mathbb{S}, \mathcal{T})$ forms a $(0, 1)$-lattice with lattice operations defined by:

$$A \wedge B = A \cap B;$$

$$A \vee B = A \cup B \cup \{y \in Y \mid f(y) = (a, b) \text{ and } a, b \in X \cap (A \cup B)\}.$$

Definition 3.4. For spectral N_5-spaces $(\mathbb{S}, \mathcal{T})$ and $(\mathbb{S}', \mathcal{T}')$ and a map $\varphi\colon \mathbb{S} \to \mathbb{S}'$ that is N_5-morphism, we say that φ is a *spectral* N_5-*morphism* if, for every N_5-compact open set A in $(\mathbb{S}', \mathcal{T}')$, the set $\varphi^{-1}(A)$ is N_5-compact open in $(\mathbb{S}, \mathcal{T})$.

The functor $\mathsf{L}\colon \mathbb{T}_5 \to \mathbb{L}_5$ is defined as follows, where $(\mathbb{S}, \mathcal{T})$, $(\mathbb{S}', \mathcal{T}')$ are objects of $\mathbb{T}_5$, and $\varphi\colon \mathbb{S} \to \mathbb{S}'$ is a spectral N_5-morphism:

$\mathsf{L}(\mathbb{S}, \mathcal{T})$ is the $(0, 1)$-lattice defined above;

$\mathsf{L}(\varphi)\colon \mathsf{L}(\mathbb{S}', \mathcal{T}') \to \mathsf{L}(\mathbb{S}, \mathcal{T})$ is given by $\mathsf{L}(\varphi)(A') = \varphi^{-1}(A')$.

Proposition 3.5. The following statements hold.

(1) $\mathsf{L}(\mathbb{S}, \mathcal{T})$ forms a lattice which is a $(0, 1)$-sublattice of the ideal lattice of $\mathbb{S}$ $(= \mathsf{B}(\mathbb{S}))$ and so $\mathsf{L}(\mathbb{S}, \mathcal{T})$ belongs to $\mathbf{SP}(N_5)$.

(2) $\mathsf{L}(\varphi)\colon \mathsf{L}(\mathbb{S}', \mathcal{T}') \to \mathsf{L}(\mathbb{S}, \mathcal{T})$ is a $(0, 1)$-lattice homomorphism.

Remark 3.6. Proof of the above proposition does not use the condition (5) of Definition 3.1.

Corollary 3.7. $\mathsf{L}\colon \mathbb{T}_5 \to \mathbb{L}_5$ is a contravariant functor.

In order to construct a contravariant functor $\mathsf{T} : \mathbb{L}_5 \to \mathbb{T}_5$, we now consider, for a $(0,1)$-lattice $L \in \mathbf{SP}(N_5)$, the lattice $F(L)$ of filters on L with the inverse inclusion as the lattice order. The lattices L and $F(L)$ satisfy the same lattice equations. Moreover, $F(L)$ is dually algebraic. This implies that every element of $F(L)$ is join-irreducible (in fact, completely join-irreducible) or is the lattice join in $F(L)$ of the join-irreducible (in fact, completely join-irreducible) elements below it. We say that a filter on L is *join-irreducible* or *join-prime* if the filter regarded as an element of $F(L)$ is join-irreducible or join-prime, respectively.

Let $X(L)$ denote the set of join-prime filters of L and let $Y(L)$ denote the set of join-irreducible filters of L which are not join-prime. Then $S(L) = X(L) \cup Y(L)$ consists of all join-irreducible filters of L. We put $\mathbb{S}(L) = \langle X(L), Y(L), \supseteq, f\rangle$, where $f\colon Y(L) \to X(L)^2$ is a function which is defined as follows: $f(F) = (G, H)$, where $\{G, H\}$ is the unique pair of elements of $X(L)$ which, by Lemma 1.1, exists for F and, by choice, $F \subset G$. One can show that $\mathbb{S}(L)$ is an N_5-space and, consequently, the ideal lattice of $\mathbb{S}(L)(= \mathsf{B}(L))$ is bi-algebraic and belongs to $\mathbf{SP}(N_5)$.

We now enhance $\mathbb{S}(L)$ by a topology and denote it by $\mathcal{T}(L)$. As a consequence, we obtain the pair $(\mathbb{S}(L), \mathcal{T}(L))$.

For $x \in L$, let $I(x) = \{F \in S(L) \mid x \in F\}$ and for $M \subseteq L$, let $I(M) = \bigcup_{x\in M} I(x)$.

Definition 3.8. The open sets of $\mathcal{T}(L)$ are exactly sets of the form $I(M)$, where $M \subseteq L$.

Remark 3.9. Notice that the collection of all sets $I(x)$, $x \in L$, is a multiplicative base for $\mathcal{T}(L)$. This is so because $I(x) \cap I(y) = I(x \wedge y)$. Notice also that $I(x)$ is f-closed in $\mathbb{S}(L)$ and that $\leq_{\mathcal{T}(L)}$ coincides with $\subseteq$ because $\mathcal{T}(L)$ is T_0. Moreover, one can show that the family $\{I(x) \mid x \in L\}$ is exactly the collection of all sets that are f-closed in $\mathbb{S}$, open in $(S(L), \mathcal{T}(L))$, and N_5-compact in $(\mathbb{S}(L), \mathcal{T}(L))$. Also, one can show that $X(L)$ with the topology $\mathcal{T}(L)$ restricted to $X(L)$ is a spectral space. And also, one can show that $(\mathbb{S}(L), \mathcal{T}(L))$ fulfills the condition (5) of Definition 3.1 according to Definition 3.10 that is now given below.

For N_5-spaces $\mathbb{S} = \langle X, Y, \leq, f\rangle$ and $\mathbb{S}' = \langle X', Y', \leq', f'\rangle$ with $X \cup Y \subseteq X' \cup Y'$ and $2X \subseteq 2X'$, we say that $\mathbb{S}$ is an N_5-*subspace* of $\mathbb{S}'$ if the identity map from $X \cup Y \cup 2X$ to $X' \cup Y' \cup 2X'$ is an N_5-morphism.

Definition 3.10. For pairs $(\mathbb{S}, \mathcal{T})$ and $(\mathbb{S}', \mathcal{T}')$ satisfying the conditions (1)–(4) of Definition 3.1 with $\mathbb{S}$ being an N_5-subspace of $\mathbb{S}'$ and $(X \cup Y, \mathcal{T})$ a topological subspace of $(X' \cup Y', \mathcal{T}')$, we say that $(\mathbb{S}', \mathcal{T}')$ is a uN_5-*extension* of $(\mathbb{S}, \mathcal{T})$ if, for every A that is f-closed in $\mathbb{S}'$, N_5-compact in $(\mathbb{S}', \mathcal{T}')$, and open in $(X' \cup Y', \mathcal{T}')$, the following holds:

$$A = \bigcup \{B \in \mathcal{T}' \mid B \cap (X \cup Y) = A\}.$$

Remark 3.11. The notion of uN_5-extension originates from the concept of u-extension considered in a general topological context in [6], see also [7] and [8].

The functor $\mathsf{T}\colon \mathbb{L}_5 \to \mathbb{T}_5$ on objects of $\mathbb{L}_5$ is defined by $\mathsf{T}(L) = (\mathbb{S}(L), \mathcal{T}(L))$ and $\mathsf{T}(g)\colon \mathsf{T}(L') \to \mathsf{T}(L)$ on morphisms $g\colon L \to L'$ of $\mathbb{L}_5$ by

$$\mathsf{T}(g)(F) = \begin{cases} g^{-1}(F), & \text{if } g^{-1}(F) \in X(L) \cup Y(L); \\ \{g^{-1}(G), g^{-1}(H)\}, & \text{if } g^{-1}(F) \in 2X(L) \text{ and } f(F) = (G, H). \end{cases}$$

Proposition 3.12. The following statements hold.

(1) $\mathsf{T}(L)$ is spectral N_5-space.

(2) $\mathsf{T}(g)\colon \mathsf{T}(L') \to \mathsf{T}(L)$ is a spectral N_5-morphism.

Corollary 3.13. $\mathsf{T}\colon \mathbb{L}_5 \to \mathbb{T}_5$ is a contravariant functor.

For $(\mathbb{S}, \mathcal{T}) \in \mathbb{T}_5$ and $L \in \mathbb{L}_5$, we define

$$\tau_{(\mathbb{S},\mathcal{T})}\colon (\mathbb{S}, \mathcal{T}) \to \mathsf{T}(\mathsf{L}(\mathbb{S}, \mathcal{T})) \text{ by } \tau_{(\mathbb{S},\mathcal{T})}(x) = \{A \in \mathsf{L}(\mathbb{S}, \mathcal{T}) \mid x \in A\};$$

$$\rho_L\colon L \to \mathsf{L}(\mathsf{T}(L)) \text{ by } \rho_L(x) = \{F \in \mathbb{S}(L) \mid x \in F\}.$$

Proposition 3.14. The following statements hold.

(1) $\tau_{(\mathbb{S},\mathcal{T})}$ is an N_5-isomorphism on the N_5-space part of $(\mathbb{S}, \mathcal{T})$ and a homeomorphism on the topological part of $(\mathbb{S}, \mathcal{T})$. Moreover, for every morphism $\varphi\colon (\mathbb{S}, \mathcal{T}) \to (\mathbb{S}', \mathcal{T}')$ in $\mathbb{T}_5$, $\mathsf{TL}(\varphi) \circ \tau_{(\mathbb{S},\mathcal{T})} = \tau_{(\mathbb{S}',\mathcal{T}')} \circ \varphi$.

(2) ρ_L is a $(0, 1)$-lattice isomorphism. Moreover, for every morphism $f\colon L \to L'$ in $\mathbb{L}_5$, $\mathsf{LT}(f) \circ \rho_L = \rho_{L'} \circ f$.

Remark 3.15. We are now ready to explain the role of the condition (5) of Definition 3.1. In the proof of (1) of Proposition 3.14, it is established first that $\tau_{(\mathbb{S},\mathcal{T})}$ is an embedding in the N_5-space and topological space sense. Next, it is established that $\mathsf{T}(\mathsf{L}(\mathbb{S},\mathcal{T}))$ is a uN_5-extension of the image of $(\mathbb{S},\mathcal{T})$ by $\tau_{(\mathbb{S},\mathcal{T})}$. This, by the condition (5) of Definition 3.1 implies that that $\tau_{(\mathbb{S},\mathcal{T})}$ is surjective.

Corollaries 3.7, 3.13 and Proposition 3.14 justify the following theorem.

THEOREM 3.16. *The categories* $\mathbb{L}_5$ *and* $\mathbb{T}_5$ *are dually equivalent.*

COROLLARY 3.17. The categories $(\mathbb{L}_5)_{fin}$ and $(\mathbb{T}_5)_{fin}$ are dually equivalent.

Remark 3.18. Under the assumption that $Y = \varnothing$ in all spectral N_5-spaces, we obtain the category of spectral spaces with spectral morphisms which, as was proved by M. H. Stone in [16], is dually equivalent to the category of bounded distributive lattices with $(0,1)$-lattice homomorphism as morphisms.

4. The Q-lattice of a non-trivial variety of bounded lattices

Let $\mathbb{P}_2$ denote the category whose objects are partially ordered sets with two distinguished constants and morphisms are mappings that preserve partial orders and the distinguished constants.

Theorem 1.5 of [4] says that the category $\mathbb{P}_2$ is universal. An inspection of the proof of this result presented in [4] shows more. It shows that $\mathbb{P}_2$ is finite-to-finite universal. This means that there is a faithful and full functor from the category of undirected graphs with all compatible maps as morphisms to the category $\mathbb{P}_2$ and has the property that it assigns a finite object of $\mathbb{P}_2$ to every finite graph. This in turn means that in the category $\mathbb{P}_2$ there exists a family of finite objects $A_i = \langle X_i, \leq_i, a_i, b_i\rangle$, $i < \omega$, which has the property:

($*$) For $i, j < \omega$, there is a morphism of $\mathbb{P}_2$ between A_i and A_j iff $i = j$.

For each $i < \omega$, let y_i be an element not belonging to X_i, and let $A_i^+ = \langle X_i, \{y_i\}, \leq_i^+, f_i\rangle$, where $\leq_i^+ = \leq_i \cup\{(a_i, y_i), (y_i, y_i)\}$ and $f_i(y_i) = (a_i, b_i)$. Each A_i^+ is a finite N_5-space. Corollary 2.11 and the property ($*$) imply the following: For $I, J \subseteq \omega$, $\mathbf{SP}(\mathsf{F}(A_i^+) \mid i \in I) = \mathbf{SP}(\mathsf{F}(A_i^+) \mid i \in J)$ iff $I = J$. Thus the Q-lattice of the variety of bounded lattices generated by N_5 is uncountable. Without much effort, one can construct a finite N_5-space $\mathbb{S}$ such that the quasivariety generated by $\mathsf{B}(\mathbb{S})$ is a join-irreducible but not join-prime element in the Q-lattice. This means that the Q-lattice of the variety $\mathbf{SP}(N_5)$ is not distributive. On the other hand, by Corollary 1.5 of [2], we know that the variety of bounded lattices generated by M_3 is uncountable and non-distributive. The two lattices N_5 and M_3 are the only lattices which separate lattices from those which are distributive. As the Q-lattice of the variety of bounded distributive lattice is a 2-element chain, the result announced in the abstract is true: The Q-lattice of any nontrivial variety of bounded lattices is either a 2-element chain or is uncountable and non-distributive.

5. Concluding remarks

Our first duality is an extension of the well known due to G. Birkhoff duality for distributive bi-algebraic lattices (assume that $Y = \varnothing$ in the definition of N_5-space). Our second duality is an extension of the Stone topological duality for bounded distributive lattices. The categories of duals of the Stone and the well-known Priestley [15] duality for bounded distributive lattices are equivalent (see [5]). Our work confirms (see also [13]) that a successful attempt of having topological dualities in the categorical (complete) sense for bounded lattices should be focused only on a variety that is generated by a finite lattice and the outcome will be in the style proposed by M. H. Stone in [16]. The key concept in searching for them will be the concept of minimal join cover refinement property and the navigating result will be Theorem 3.4 of [14]. Our original motivation for having the first duality was the open problem independently raised by G. Birkhoff and A. I. Maltsev which asks for a description of the Q-lattices (see [11] or the survey article [1]). Based on our experience, we know that having a good duality helps in contributing to this open problem. However, we do not know what is the real lattice status of the Q-lattice of the variety of bounded lattices generated by N_5. For example, does this Q-lattice satisfy any non-trivial lattice equation?

A result of [10] states that a variety of bounded lattices is universal iff it contains a non-distributive simple lattice. Moreover, it states that the variety is finite-to-finite universal iff the simple lattice is finite. As M_3 is a simple lattice that is not distributive, the variety of bounded lattices generated by M_3 is finite-to-finite universal. The variety of bounded lattices generated by N_5 is not universal for N_5 is not a simple lattice. Our original motivation of having the second duality was to know an answer to the following question: Is the variety of bounded lattices generated by N_5 finite-to-finite universal relative to the variety of bounded distributive lattices? The *relative* means that $(0, 1)$-lattice homomorphisms to bounded distributive lattices are disregarded in the successful construction of a functor from the category of undirected graphs to the category of bounded lattices generated by N_5. We do not know an answer to this well coined by literature question.

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References

[1] M. E. Adams, K. V. Adaricheva, W. Dziobiak, A. V. Kravchenko, *Open questions related to the problem of Birkhoff and Maltsev*, **Studia Logica**, vol. 78 (2004), pp. 357–378, DOI: https://doi.org/10.1007/s11225-005-7378-x.

[2] M. E. Adams, W. Dziobiak, *Finite-to-finite universal quasivarieties are Q-universal*, **Algebra Universalis**, vol. 46 (2001), pp. 253–283, DOI: https://doi.org/10.1007/PL00000343.

[3] M. E. Adams, W. Dziobiak, A. V. Kravchenko, M. V. Schwidefsky, *Complete homomorphic images of the quasivariety lattices of locally finite quasivarieties* (2020).

[4] M. E. Adams, V. Koubek, J. Sichler, *Homomorphisms and endomorphisms of distributive lattices*, **Houston Journal of Mathematics**, vol. 11 (1984), pp. 129–145, DOI: https://doi.org/10.2307/1999472.

[5] W. H. Cornish, *On H. Priestley's dual of the category of bounded distributive lattices*, **Matematički Vesnik**, vol. 12 (1975), pp. 329–332.

[6] Y. L. Ershov, *Solimit points and u-extensions*, **Algebra and Logic**, vol. 56 (2017), pp. 295–301, DOI: https://doi.org/10.1007/s10469-017-9450-9.

[7] Y. L. Ershov, M. V. Schwidefsky, *To the spectral theory of partially ordered sets*, **Siberian Mathematical Journal**, vol. 60 (2019), pp. 450–463, DOI: https://doi.org/10.1134/S003744661903008X.

[8] Y. L. Ershov, M. V. Schwidefsky, *To the spectral theory of partially ordered sets. II*, **Siberian Mathematical Journal**, vol. 61 (2020), pp. 453–462, DOI: https://doi.org/10.1134/S0037446620030064.

[9] R. Freese, J. B. Nation, J. Ježek, **Free Lattices**, no. 42 in Mathematical Surveys and Monographs, American Mathematical Society, Providence (1995).

[10] P. Goralčík, V. Koubek, J. Sichler, *Universal varieties of (0,1)-lattices*, **Canadian Journal of Mathematics**, vol. 42 (1990), pp. 470–490, DOI: https://doi.org/10.4153/CJM-1990-024-0.

[11] V. A. Gorbunov, **Algebraic Theory of Quasivarieties**, Siberian School of Algebra and Logic, Plenum, Consultants Bureau, New York (1998).

[12] A. P. Huhn, *Schwach distributive Verbände. I*, **Acta Scientiarum Mathematicarum (Szeged)**, vol. 33 (1972), pp. 297–305.

[13] M. A. Moshier, P. Jipsen, *Topological duality and lattice expansions, I: A topological construction for canonical extensions*, **Algebra Universalis**, vol. 71 (2014), pp. 109–126, DOI: https://doi.org/10.1007/s00012-014-0275-2.

[14] J. B. Nation, *An approach to lattice varieties of finite height*, **Algebra Universalis**, vol. 27 (1990), pp. 521–543, DOI: https://doi.org/10.1007/BF01188998.

[15] H. A. Priestley, *Ordered topological spaces and representation of distributive lattices*, **Proceedings of the London Mathematical Society**, vol. 24 (1972), pp. 507–530, DOI: https://doi.org/10.1112/plms/s3-24.3.507.

[16] M. H. Stone, *Topological representation of distributive lattices and Brouwerian logics*, **Časopis pro pěstování mathematiky a fysiky**, vol. 67 (1938), pp. 1–25.

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