FUNDAMENTAL RELATION ON $H_v$BE-ALGEBRAS

Abstract

In this paper, we are going to introduce a fundamental relation on $H_v$BE-algebra and investigate some of its properties, also construct new $(H_v)BE$-algebras via this relation. We show that quotient of any $H_v$BE-algebra via a regular regulation is an $H_v$BE-algebra and this quotient, via any strongly relation is a $BE$-algebra. Furthermore, we investigate that under what conditions some relations on $H_v$BE-algebra are transitive relations.

Keywords: $(H_v, Hyper)BE$-algebra, fundamental relation, quotient.

2020 Mathematical Subject Classification: 06F35, 03G25.

1. Introduction and preliminaries

Hyperstructures represent a natural extension of classical structures and they were introduced by French Mathematician F. Marty in 1934 [10]. A hyperstructure is a nonempty set $H$, together with a function $\circ : H \times H \rightarrow P^*(H)$ called hyper operation, where $P^*(H)$ denotes the set of all nonempty subsets of $H$. Marty introduced hypergroups as a generalization of groups [4, 3]. Hyperstructures have many applications to several sectors of both pure and applied sciences as geometry, graphs and hypergraphs, fuzzy sets and rough sets, automata, cryptography, codes, relation algebras, C-algebras, artificial intelligence, probabilities, chemistry, physics, especially in atomic physics and in harmonic analysis [2, 7].
H. S. Kim and Y. H. Kim introduced the notation of BE-algebra as a generalization of dual BCK algebra [9]. R. A. Borzooei et al. defined the notation of a hyper K-algebra, bounded hyper K-algebra and considered the zero condition in hyper K-algebras. They showed that every hyper K-algebra with the zero condition can be extended to a bounded hyper K-algebra [1].

A. Radfar et al. combined BE-algebra with hyperstructures and defined the notation of hyper BE-algebra. Also, they focused on some types of hyper BE-algebras and show that every dual hyper K-algebra is a hyper BE-algebra [11].

We know that the class of the Hv-structures, introduced by Vougiouklis in 1990 [13, 14], is the largest class of hyperstructures. In the classical hyperstructures, in any axiom where the equality is used, if we replace the equality by the nonempty intersection, then we obtain a corresponding Hv-structures.

F. Iranmanesh et al. present the notation of the HvBE-algebra as generalization of hyper BE-algebra [8]. They defined new Hv-structures and considered some of their useful properties. Also discuss Hv-filters and homomorphism on this structure. Furthermore, they got more results in HvBE-algebras [8]. Fundamental relations are one of the main tools in algebraic hyperstructures theory.

Algebraic hyperstructures are extension of algebraic structures and for better understanding their properties we want some connections between algebraic hyperstructures and algebraic structures, a fundamental relation is an interesting concept in algebraic hyperstructures that makes this connection. In this paper, for obtain a relationship between BE, hyper BE and (Hv)BE-algebra, we define a fundamental relation on (Hv)BE-algebra that is called “δ” also, we study “δ∗” as a transitive closure of “δ” in such away that is the smallest equivalence relation that contains “δ”. Finally, a BE-algebras which is quotient of HvBE-algebra via “δ∗” is obtained, therefore we find a connection between algebraic structures and (Hv)hyper algebraic structures.

**Definition 1.1 ([9]).** Let X be a nonempty set, “∗” be a binary operation on X and a constant 0 ∈ X. Then (X, ∗, 0) is called a BCK-algebra if for all x, y, z ∈ X it satisfies the following conditions:

(BCI-1) \(((x ∗ y) ∗ (x ∗ z)) ∗ (z ∗ y) = 0,\)
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(BCI-2) $(x * (x * y)) * y = 0,$
(BCI-3) $x * x = 0,$
(BCI-4) $x * y = 0$ and $y * x = 0,$ imply $x = y,$
(BCI-5) $0 * x = 0.$

We define a binary relation “$\leq$” on $X$ by $x \leq y$ if and only if $x * y = 0.$

**Definition 1.2 ([9]).** Let $X$ be a nonempty set, “*” be a binary operation on $X$ and $1 \in X.$ An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if the following axioms hold:

(BE1) $x * x = 1,$
(BE2) $x * 1 = 1,$
(BE3) $1 * x = x,$
(BE4) $x * (y * z) = y * (x * z),$ for all $x, y, z \in X.$

We introduce the relation “$\leq$” on $X$ by $x \leq y$ if and only if $x * y = 1.$ The BE-algebra $(X, *, 1)$ is said to be commutative, if for all $x, y \in X,$

$$(x * y) * y = (y * x) * x.$$  

**Proposition 1.3 ([9]).** Let $X$ be a BE-algebra. Then

(i) $x * (y * x) = 1.$
(ii) $y * ((y * x) * x) = 1,$ for all $x, y \in X.$

**Example 1.4 ([12]).** Let $X = \{1, 2, \ldots\}.$ Define the operation “*” as follows:

$$x * y = \begin{cases} 1 & \text{if } y \leq x \\ y & \text{otherwise.} \end{cases}$$

then $(X, *, 1)$ is a BE-algebra.

**Definition 1.5 ([4]).** Let $H$ be a nonempty set and $\circ : H \times H \rightarrow P^*(H)$ be a hyperoperation. Then $(H, \circ)$ is called an $H_v$-group if it satisfies the following axioms:

(H1) $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset,$
(H2) $a \circ H = H \circ a = H,$ for all $x, y, z, a \in H.$
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where \( a \circ H = \bigcup_{h \in H} a \circ h \), \( H \circ a = \bigcup_{h \in H} h \circ a \).

**Definition 1.6 ([11])**. Let \( H \) be a nonempty set and \( \circ : H \times H \rightarrow P^*(H) \) be a hyperoperation. Then \((H, \circ, 0)\) is called a hyper \( K \)-algebra if satisfies the following axioms:

- \( (HK1) \) \( (x \circ z) \circ (y \circ z) < x \circ y \),
- \( (HK2) \) \( (x \circ y) \circ z = (x \circ z) \circ y \),
- \( (HK3) \) \( x < x \),
- \( (HK4) \) \( x < y \) and \( y < x \) implies \( x = y \),
- \( (HK5) \) \( 0 < x \), for all \( x, y, z \in H \),

where the relation "\(<\)" is defined by \( x < y \iff 0 \in x \circ y \). For every \( A, B \subseteq H \), \( A < B \) if and only if there exist \( a \in A \) and \( b \in B \) such that \( a < b \). Note that if \( A, B \subseteq H \), then by \( A \circ B \) we mean the subset \( \bigcup_{a \in A, b \in B} a \circ b \) of \( H \).

**Definition 1.7 ([11])**. Let \( H \) be a nonempty set and \( \circ : H \times H \rightarrow P^*(H) \) be a hyperoperation. Then \((H, \circ, 1)\) is called a hyper \( BE \)-algebra if satisfies the following axioms:

- \( (HBE1) \) \( x < 1 \) and \( x < x \),
- \( (HBE2) \) \( x \circ (y \circ z) = y \circ (x \circ z) \),
- \( (HBE3) \) \( x \in 1 \circ x \),
- \( (HBE4) \) \( 1 < x \) implies \( x = 1 \), for all \( x, y, z \in H \).

\((H, \circ, 1)\) is called a dual hyper \( K \)-algebra if it satisfies \((HBE1), (HBE2)\) and the following axioms:

- \( (DHK1) \) \( x \circ y < (y \circ z) \circ (x \circ z) \),
- \( (DHK4) \) \( x < y \) and \( y < x \) imply that \( x = y \),

where the relation "\(<\)" is defined by \( x < y \iff 1 \in x \circ y \).
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**Definition 1.8 ([8]).** Let $H$ be a nonempty set and $\circ : H \times H \rightarrow P^*(H)$ be a hyperoperation. Then $(H, \circ, 1)$ is called an $H_v$BE-algebra if satisfies the following axioms:

- $(H_vBE1)$ $x < 1$ and $x < x$,
- $(H_vBE2)$ $x \circ (y \circ z) \cap y \circ (x \circ z) \neq \phi$,
- $(H_vBE3)$ $(H_vBE3)$ $x \in 1 \circ x$,
- $(H_vBE4)$ $1 < x$ implies $x = 1$, for all $x, y, z \in H$,

where the relation “$<$” is defined by $x < y \iff 1 \in x \circ y$.

Also $A < B$ if and only if there exist $a \in A$ and $b \in B$ such that $a < b$.

**Proposition 1.9 ([6]).** Every dual hyper $K$-algebra is a hyper $BE$-algebra.

2. **On $H_v$BE-algebras and some results**

In this section, we consider $H_v$BE-structure with some results on its.

**Example 2.1.**

(i) Let $(H, *, 1)$ be a $BE$-algebra and we know that $\circ : H \times H \rightarrow P^*(H)$ with $x \circ y = \{x * y\}$ is a hyperoperation. Then $(H, \circ, 1)$ is a trivial hyper $BE$-algebra and an $H_v$BE-algebra.

(ii) Let $H = \{1,a,b\}$. Define hyperoperation “$\circ$” as follows:

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Then $(H, \circ, 1)$ is an $H_v$BE-algebra.

(iii) Define the hyperoperation “$\circ$” on $\Bbb{R}$ as follows:

$$x \circ y = \begin{cases} \{y\} & \text{if } x = 1 \\ \Bbb{R} & \text{otherwise,} \end{cases}$$

then $(\Bbb{R}, \circ, 1)$ is an $H_v$BE-algebra.
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**Proposition 2.2** ([8]). Any hyper $BE$-algebra is an $H_vBE$-algebra.

Example 2.3 shows that the converse of Proposition 2.2 does not hold in general.

**Example 2.3.** Define a hyperoperation "\(\circ\)" on the set \(H = \{1, a, b\}\) as follows:

\[
\begin{array}{c|ccc}
\circ & 1 & a & b \\
\hline
1 & \{1\} & \{a\} & \{b\} \\
 a & \{1, b\} & \{1\} & \{1, a, b\} \\
 b & \{1\} & \{1, b\} & \{1, b\} \\
\end{array}
\]

Then \((H, \circ, 1)\) is an $H_vBE$-algebra. And we have that:

\[a \circ (b \circ b) = a \circ (\{1, b\}) = \{1, a, b\} \neq \{1, b\} = b \circ (\{1, a, b\}) = b \circ (a \circ b)\]

So \((H, \circ, 1)\) does not satisfy \((HBE)\), therefore \((H, \circ, 1)\) is not a hyper $BE$-algebra.

**Theorem 2.4.** Let \((H, \circ, 1)\) be an $H_vBE$-algebra. Then

(i) \(A \circ (B \circ C) \cap B \circ (A \circ C) \neq \phi\) for every \(A, B, C \in P^*(H)\),

(ii) \(A < A\),

(iii) \(1 < A\) implies \(1 \in A\),

(iv) \(1 \in x \circ (x \circ x)\),

(v) \(x < x \circ x\).

**Proof:** (i) Let \(a \in A, b \in B\) and \(c \in C\). We have \(a \circ (b \circ c) \subseteq A \circ (B \circ C), b \circ (a \circ c) \subseteq B \circ (A \circ C)\). Then by \((H_vBE)\), \(a \circ (b \circ c) \cap b \circ (a \circ c) \neq \phi\), therefore \(A \circ (B \circ C) \cap B \circ (A \circ C) \neq \phi\).

Other cases are similar. \(\square\)

**Definition 2.5** ([6]). Let \(F\) be a nonempty subset of $H_vBE$-algebra \(H\) and \(1 \in F\). Then \(F\) is called

(i) a weak $H_v$-filter of \(H\) if \(x \circ y \subseteq F\) and \(x \in F\) imply \(y \in F\), for all \(x, y \in H\).

(ii) an $H_v$-filter of \(H\) if \(x \circ y \approx F\) (i.e. \(\phi \neq (x \circ y) \cap F\)) and \(x \in F\) imply \(y \in F\), for all \(x, y \in H\).
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**Example 2.6.** Let $H = \{1,a,b\}$. Define the hyperoperation “$\circ_1$” and “$\circ_2$” as follows:

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<td>${a,b}$</td>
<td>${b}$</td>
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<tr>
<td>a</td>
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<td>${1,a}$</td>
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<td>${1,a,b}$</td>
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<table>
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<tr>
<th>$\circ_2$</th>
<th>1</th>
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<td>${1}$</td>
<td>${a,b}$</td>
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<td>b</td>
<td>${1}$</td>
<td>${1,a,b}$</td>
<td>${1,a,b}$</td>
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We see that $(H, \circ_1, 1)$ is an $H_v$BE-algebra and $F_1 = \{1,a\}$ is a weak $H_v$-filter of $H$. Also $(H, \circ_2, 1)$ is an $H_v$BE-algebra and $F_2 = \{1,a\}$ is an $H_v$-filter of $H$.

In Example 2.6, $F_1$ is not an $H_v$-filter, because $a \circ_1 b \not\approx F_1$ and $a \in F_1$, but $b \not\in F_1$.

**Theorem 2.7.** Every $H_v$-filter is a weak $H_v$-filter.

**Notation.** By Example 2.6, we can see that the notion of weak $H_v$-filter and $H_v$-filter are different in $H_v$BE-algebra.

**Theorem 2.8 ([8]).** Let $F$ be a subset of an $H_v$BE-algebra $H$ and $1 \in F$. For all $x, y \in H$, if $x \circ y < F$ and $x \in F$ implies $y \in F$, then $F = H$.

### 3. Relations on $H_v$BE-algebras

In this section, let $(H, \circ, 1)$ be a $H_v$BE-algebra and presents in summary with $H$. We show that there exists a connection between hyper algebraic structures and algebraic structures by strongly regular relations.

**Definition 3.1.** Let $(H, \circ, 1)$ be an $H_v$BE-algebra and $R$ be an equivalence relation on $H$. If $A, B$ are nonempty subsets of $H$, then

(i) $A \bar{R} B$ means that, for all $a \in A$, there exists $b \in B$ in such away that $aRb$ and for all $b' \in B$, there exists $a' \in A$ in such away that $b'Ra'$,

(ii) $A \bar{B} B$ means that, for all $a \in A$ and $b \in B$, we have $aRb$,

(iii) $R$ is called right regular(left regular), if for all $x \in H$, from $aRb$, it follows that $(a \circ x) \bar{R}(b \circ x)((x \circ a) \bar{R}(x \circ b)).$

(iv) $R$ is called strongly right regular(strongly left regular), if for all $x \in H$, from $aRb$, it follows that $(a \circ x) \bar{R}(b \circ x)((x \circ a) \bar{R}(x \circ b)).$
(v) $R$ is called (strongly) regular, if it is (strongly) right regular and (strongly) left regular,

(vi) $R$ is called good, if \((a \circ b) \, R \, 1 \) and \((b \circ a) \, R \, 1 \) imply \(aRb\), for all \(a, b \in H\).

It is clear that \((a \circ b) \, R \, 1\) means that there exists \(x \in a \circ b\) in such a way that \(xR1\).

**Example 3.2.** Let \(H = \{1, a, b\}\). Define the hyperoperation “\(\circ\)” as follows:

\[
\begin{array}{ccc}
\circ & 1 & a & b \\
\hline
1 & \{1\} & \{a, b\} & \{b\} \\
 a & \{1\} & \{1, a, b\} & \{b\} \\
 b & \{1, b\} & \{1, a, b\} & \{1, a, b\}
\end{array}
\]

Then \((H, \circ, 1)\) is an \(H, BE\)-algebra. It is easy to see that

\[R = \{(1, 1), (a, a), (b, b), (a, b), (b, a), (1, b), (b, 1), (a, 1), (1, a)\}\]

is a good strongly regular relation on \(H\) and for any \(A, B \in P^*(H)\), \(A \bar{R} B\).

**Example 3.3.** Let \(H = \{1, d, b, c\}\). Define the hyperoperation “\(\circ\)” as follows:

\[
\begin{array}{cccc}
\circ & 1 & b & c & d \\
\hline
1 & \{1\} & \{b\} & \{c\} & \{d\} \\
 b & \{1\} & \{1\} & \{1\} & \{1\} \\
 c & \{1\} & \{b\} & \{1\} & \{d\} \\
 d & \{1\} & \{b\} & \{1, c\} & \{1\}
\end{array}
\]

Then \((H, \circ, 1)\) is an \(H, BE\)-algebra. It is easy to see that

\[R = \{(1, 1), (d, d), (b, b), (c, c), (c, b), (b, c), (d, c), (c, d)\}\]

is not regular and strongly regular relation on \(H\).

**Notation.** Let \(R\) be regular relation on \(H\). We denote the set of all equivalence classes of \(R\) by \(H/R\). Hence \(H/R = \{\bar{x} : x \in H\}\). For any \(\bar{x}, \bar{y} \in H/R\), define a hyperoperation “\(*\)” on \(H/R\) by

\[\bar{x} \ast \bar{y} = \{\bar{z} : z \in x \circ y\}\]

and a binary relation “\(<\)” on \(H/R\) by
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"$\bar{x} < \bar{y}$" $\iff \bar{1} \in \bar{x} \ast \bar{y}$.

**Lemma 3.4.** Let $R$ be a regular relation on $H$. Then $(H/R; \ast)$ is a hypergroupoid.

**Proof:** We must show that $\ast$ be well defined. Let $x_1 \ast y_1 \ast y_2 \ast y_3 \in H/R$ such that $x_1 \ast y_1 = x_2 \ast y_2$. Then $x_1 \ast R \ast y_1 \ast R \ast y_2$. Since $R$ is a regular relation, we have $(x_1 \ast y_1) \ast R(x_2 \ast y_2)$ [5]. Let $\bar{r} \in x_1 \ast y_1$. Then there exists $z \in x_1 \ast y_1$ in such a way that $\bar{r} = \bar{z}$. Now $z \in x_1 \ast y_1 \ast y_2$ and $(x_1 \ast y_1) \ast R(x_2 \ast y_2)$, then there exists $u \in (x_2 \ast y_2)$ such that $zRu$ then $\bar{z} = \bar{u}$ and $\bar{r} = \bar{u}$, thus $x_1 \ast y_1 \ast y_2 \ast y_3 \ast x_1 \ast y_1 \ast y_2$ and in a similar way we get $x_1 \ast y_1 \ast y_2 \ast y_3 \ast x_1 \ast y_1 \ast y_2$, i.e. $x_1 \ast y_1 \ast y_2 \ast y_3 \ast x_1 \ast y_1 \ast y_2$, therefore $\ast$ is well defined and $(H/R; \ast)$ is a hypergroupoid. 

**Theorem 3.5.** If $R$ is a regular relation on $H$ then $(H/R; \ast; \bar{1})$ is a $H_v$BE-algebra.

**Proof:** Let $R$ be a regular relation on $H$. If $x \in H$ then $\bar{x} \ast \bar{1} = \{\bar{t} : t \in x \circ 1\}$. Since $H$ is an $H_v$BE-algebra by $(H_v, BE1)$ we conclude that $1 \in x \circ 1$ and so $\bar{1} \in \bar{x} \ast \bar{1}$. Therefore $\bar{x} < \bar{1}$. Also $1 \in x \circ x$ and $\bar{x} \circ \bar{x} = \{\bar{t} : t \in x \circ x\}$, then $\bar{1} \in \bar{x} \ast \bar{x}$ and $\bar{x} < \bar{1}$.

$(H_v, BE2)$ Let $x, y, z \in H$. Since $(H, \circ, 1)$ is an $H_v$BE-algebra, then $x \circ (y \circ z) \截 (y \circ (x \circ z)) \neq \phi$. If $t \in x \circ (y \circ z) \截 (y \circ (x \circ z))$, then there exists $s_1 \in y \circ z$ in such a way that $t \in x \circ s_1$ by a similar way there exists $s_2 \in x \circ z$ in such away that $t \in y \circ s_2$. We get the $\bar{t} \in \bar{x} \ast \bar{s}_1 \subseteq \bar{x} \ast (y \ast \bar{z})$ and $\bar{t} \in \bar{y} \ast \bar{s}_2 \subseteq \bar{y} \ast (\bar{x} \ast \bar{z})$. Therefore $\bar{x} \ast (\bar{y} \ast \bar{z}) \截 \bar{y} \ast (\bar{x} \ast \bar{z})$. Then $z \ast (\bar{y} \ast \bar{z}) \subseteq \bar{y} \ast (\bar{x} \ast \bar{z})$.

$(H_v, BE3)$ if $x \in H$ then $1 \circ \bar{x} = \{\bar{t} : t \in 1 \circ x\}$. Since $H$ is a $H_v$BE-algebra, we have $x \in 1 \circ x$ and $\bar{x} \in \bar{1} \ast \bar{x}$.

$(H_v, BE4)$ $x \in H$ and $\bar{1} \ast \bar{x}$ then $\bar{1} \in \bar{1} \ast \bar{x}$. Hence $1 \in 1 \circ x$ and $1 \in x$. Since $H$ is a $H_v$BE-algebra, we have $x = 1$ and so $\bar{x} = 1$.

**Corollary 3.6.** Let $(H, \circ, 1)$ be a dual hyper $K$-algebra and $R$ be an equivalence relation on $H$. If $R$ is a regular relation on $H$, then $(H/R; \ast; \bar{1})$ is an $H_v$BE-algebra.

**Theorem 3.7.** If $R$ is strongly regular relation on $H$, then $(H/R; \ast; \bar{1})$ is a BE-algebra.

**Proof:** If $z_1, z_2 \in \bar{x} \ast \bar{y}$, for any $x, y \in H/R$, then $z_1, z_2 \in x \circ y$. Since $R$ is strongly regular, for all $x, y \in H$, $yRy$ then $(x \circ y)\overline{R}(x \circ y)$ and $z_1, z_2 \in x \circ y$. 

we have $z_1 R z_2$, therefore $\bar{z}_1 = \bar{z}_2$ and $|\bar{x} \ast \bar{y}| = 1$ and so by Theorem 3.5, $(H/R; \ast; \bar{1})$ is a $BE$-algebra.

**Example 3.8.** Let $H = \{1, a, b, c, d, e\}$. Define the hyperoperation “$\circ$” as follows:

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Then $(H, \circ, 1)$ is an $H_vBE$-algebra. It is easy to see that $R = \{(1, 1), (a, a), (b, b), (c, c), (d, d), (e, e), (1, c), (e, b), (b, e), (a, d), (d, a)\}$ is a good strongly regular relation on $H$ and

$$H/R = \{\{1, c\}, \{a, d\}, \{e, b\}\} = \{R(1), R(a), R(b)\}.$$  

Now we have:

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<th>$R(1)$</th>
<th>$R(a)$</th>
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<td>$R(1)$</td>
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<td>$R(1)$</td>
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<td>$R(b)$</td>
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</table>

Clearly, $(H/R; \ast; R(1))$ is a $BE$-algebra.

In this place, we present some results and examples about dual hyper K-algebras and hyper BE-algebras that are useful.

**Lemma 3.9 ([6]).** Let $(X; \circ, 1)$ be a dual hyper K-algebra and $R$ be a regular relation on $X$. Then for any $\bar{x}, \bar{y}, \bar{z} \in X/R$, $(\bar{x} \ast \bar{y}) < (\bar{y} \ast \bar{z}) \ast (\bar{x} \ast \bar{z})$.

**Theorem 3.10 ([6]).** Let $(X; \circ, 1)$ be a dual hyper K-algebra and $R$ be a regular relation on $X$. If $R$ is a good relation, then $(X/R; \ast, \bar{1})$ is a dual hyper K-algebra.

**Theorem 3.11 ([6]).** Let $(X; \circ, 1)$ be a dual hyper K-algebra and $R$ be a strongly regular relation on $X$. If $R$ is a good relation, then $(X/R; \ast, \bar{1})$ is a dual BCK-algebra.
**Fundamental relation on $H_v$BE-Algebras**

**Example 3.12.** Let $X = \{1, a, b, c, d, e\}$. Define the hyperoperation “$\circ$” as follows:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
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<tbody>
<tr>
<td>1</td>
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<td>c</td>
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<td>e</td>
<td>{1, e}</td>
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</table>

Then $(X, \circ, 1)$ is a dual hyper K-algebra ($H_v$BE-algebra). It is easy to see that $R = \{(1,1),(a,a),(b,b),(c,c),(d,d),(e,e),(1,c),(c,1),(e,c),(c,e)\}$ is a good strongly regular relation on $X$ and

$X/R = \{\{1, e\}, \{a\}, \{b\}, \{c\}, \{d\}\} = \{R(1), R(a), R(b), R(c), R(d)\}$.

Now we have:

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<tr>
<th>$*$</th>
<th>$R(1)$</th>
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</table>

Clearly $(X/R; *, R(1))$ is a dual BCK-algebra.

**4. $\delta$- relation on $H_v$BE-algebra**

Let $(H; \circ, 1)$ be a $H_v$BE-algebra and $A$ be a subset of $H$. The set of all finite combinations of $A$ with hyperoperation $\circ$ and $\bigcirc_{i=1}^n a_{i\circ} = a_1 \circ a_2 \circ ... a_n$, is denoted by $L(A)$ [5].

**Definition 4.1.** Let $(H; \circ, 1)$ be a $H_v$BE-algebra. Consider:

$$\delta_1 = \{(x, x) : x \in H\}$$

and for every natural number $n \geq 1$, $\delta_n$ is defined as follows:
Farzad Iranmanesh, Mansoor Ghadiri, Arsham Borumand Saeid

\[ x \delta_n y \iff \exists (a_1, a_2, \ldots, a_n) \in H^n, \exists u \in L(a_1, a_2, \ldots, a_n) \text{ such that } \{x, y\} \subseteq u. \]

Obviously for every \( n \geq 1 \) the relations \( \delta_n \) are symmetric and no reflexive and transitive, but the relation \( \delta = \bigcup_{n \geq 1} \delta_n \) is a reflexive and symmetric relation. Let \( \delta^* \) be transitive closure of \( \delta \) (the smallest transitive relation such that contains \( \delta \)).

In the following theorem we show that \( \delta^* \) is a strongly regular relation.

**Example 4.2.** Let \( H = \{1, a, b\} \). Define the hyperoperation \( \circ \) as follows:

\[
\begin{array}{c|ccc}
\circ & 1 & a & b \\
\hline
1 & \{1\} & \{a, b\} & \{b\} \\
a & \{1\} & \{1, a, b\} & \{b\} \\
b & \{1, b\} & \{1, a, b\} & \{1, a, b\}
\end{array}
\]

Then \( (H, \circ, 1) \) is an \( H_vBE \)-algebra. \( \delta_1 = \{(x, x) : x \in H\} = \{(1, 1), (a, a), (b, b)\} \).

Since \( \{1, a\}, \{1, b\}, \{a, b\} \subseteq b \circ a \) then \( 1 \delta_2 a, 1 \delta_2 b, a \delta_2 b \). Also, we know that \( \{1, a\} \subseteq (1 \circ a) \circ b = \bigcup_{x \in 1 \circ a} (x \circ b) \) therefore \( 1 \delta_3 a \).

Similarly, \( 1 \delta_3 b, a \delta_3 b \). Obviously, \( 1 \delta_n a, 1 \delta_n b \) and \( a \delta_n b \), since \( \delta = \bigcup_{n \geq 1} \delta_n \), then \( 1 \delta a, 1 \delta b \) and \( a \delta b \).

**Theorem 4.3.** Let \( (H, \circ, 1) \) be a \( H_vBE \)-algebra. Then \( \delta^* \) is a strongly regular relation on \( H \).

**Proof:** Let \( x, y \in H \) and \( x \ \delta^* \ y \). Then we show that for any \( s \in H \) :

\[ (x \circ s) \delta^* (y \circ s). \]

Since \( \delta = \bigcup_{n \geq 1} \delta_n \) and \( \delta^* \) is the smallest transitive relation such that contains \( \delta \), then there exist \( a_0, a_1, \ldots, a_n \in H \) such that \( a_0 = x, a_n = y \) and there exist \( q_1, q_2, \ldots, q_n \in \mathbb{N} \) such that

\[ x = a_0 \delta_{q_1} a_1 \delta_{q_2} a_2 \ldots a_{n-1} \delta_{q_n} a_n = y, \]

where \( n \in \mathbb{N} \). Since for any \( 1 \leq i \leq n, a_{i-1} \ \delta_{q_i} \ a_n \), then there exists \( z_i^{+} \in H \) such that

\[ \{a_i, a_{i+1}\} \subseteq z_i^{+1} \circ z_i^{+1} \circ \ldots \circ z_i^{+1}, \]
Fundamental relation on \( H_e \)-BE-Algebras

where for \( 1 \leq m \leq n - 1 \), we have \( 1 \leq t \leq q_m \), and \( 1 \leq j \leq n - 1 \). Now, since \( s \in H \), then for all \( 0 \leq i \leq n - 1 \),

\[
a_i \circ s \subseteq z_1^{i+1} \circ z_2^{i+1} \circ \ldots \circ z_{q_i+1}^{i+1} \circ s.
\]

In a similar way, we get that

\[
a_{i+1} \circ s \subseteq z_1^{i+1} \circ z_2^{i+1} \circ \ldots \circ z_{q_i+1}^{i+1} \circ s.
\]

Then for all \( 1 \leq i \leq n \), and for all \( u \in a_i \circ s, v \in a_{i+1} \circ s \), We have \( \{u, v\} \subseteq z_1^{i+1} \circ z_2^{i+1} \circ \ldots \circ z_{q_i+1}^{i+1} \circ s \). Therefore \( u \delta_{q_i+1} v \), and so for all \( z \in a_0 \circ s = x \circ s, w \in a_n \circ s = y \circ s \), We have \( z \delta^* w \). Then \( \delta^* \) is a strongly right regular and similarly is a strongly left regular relation, therefore \( \delta^* \) is a strongly regular relation on \( H \).

**Corollary 4.4.** Let \((H, \circ, 1)\) be a hyper BE-algebra. Then \( \delta^* \) is a strongly regular relation on \( H \).

**Theorem 4.5.** Let \((H, \circ, 1)\) be a \( H_e \)-BE-algebra. \((H/\delta^*; *, \bar{1})\) is a BE algebra.

**Proof:** By Theorem 3.7 and 4.3, the proof is obvious. \( \square \)

**Example 4.6.** Let \( H = \{1, x, y, z, t\} \). Define hyperoperation “\( \circ \)” as follows:

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<td>( t )</td>
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Then \((H, \circ, 1)\) is a \( H_e \)-BE-algebra. We have \((x \circ y) \circ x = \{1, x, t\}\), \((x \circ y) \circ y = \{1, y, t\}\), \((x \circ y) \circ t = \{1, t\}\), \((x \circ y) \circ z = \{1, z, t\}\). Then for any \( u \in H, 1 \delta^* u \) and so \( \delta^* (1) = \{u \in H : 1 \delta^* u\} = H = \delta^*(1) \). Therefore \( H/\delta^* = \{\delta^*(1)\} \) and we see that \((H/\delta^*; *, \delta^*(1))\) is a trivial \( BE \)-algebra.

**Example 4.7.** Let \( H = \{1, x, y, z\} \). Define hyperoperation “\( \circ \)” as follows:

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</table>
Then \((H, \circ, 1)\) is a \(H_vBE\)-algebra. We conclude that \(H/\delta^* = \{\{1, y\}, \{x\}, \{z\}\} = \{\delta^*(1), \delta^*(x), \delta^*(z)\}\) and then:

\[
\begin{array}{c|ccc}
* & \delta^*(1) & \delta^*(x) & \delta^*(z) \\
\hline
\delta^*(1) & \delta^*(1) & \delta^*(x) & \delta^*(z) \\
\delta^*(x) & \delta^*(1) & \delta^*(1) & \delta^*(1) \\
\delta^*(z) & \delta^*(1) & \delta^*(x) & \delta^*(x) \\
\end{array}
\]

Now, by Theorem 4.3, \((H/\delta^*; *, \delta(1))\) is a \(BE\)-algebra.

**Notation.** We know that \(\delta\) is reflexive and symmetric but is not transitive on \(H\). If \(R\) is an equivalence relation on \(H\), then \(H/R\) is defined and we have the following theorem;

**Theorem 4.8 ([6]).** Let \((H, \circ, 1)\) be a hyper \(BE\)-algebra and \(R\) be an equivalence relation on \(H\). Then, \(R\) is a regular relation on \(H\) if and only if \((H/R; *, \bar{1})\) is a hyper \(BE\) algebra.

**Definition 4.9.** Let \(M\) be a nonempty subset of \(H\). \(M\) is called a \(\delta\)-part if for any \(n \in \mathbb{N}\), \(a_i \in H\), and \(L(a_1, a_2, \ldots, a_n) \cap M \neq \emptyset\), then \(L(a_1, a_2, \ldots, a_n) \subseteq M\).

**Example 4.10.** Let \(H = \{1, x, y, z\}\). Define hyperoperation “\(\circ\)” as follows:

\[
\begin{array}{c|cccc}
\circ & 1 & x & y & z \\
\hline
1 & \{1, x\} & \{1, x\} & \{y\} & \{z\} \\
x & \{1, x\} & \{1, x\} & \{y\} & \{z\} \\
y & \{1, x\} & \{1, x\} & \{1, x\} & \{z\} \\
z & \{1, x\} & \{1, x\} & \{1, x\} & \{1, x\} \\
\end{array}
\]

Then \((H, \circ, 1)\) is a \(H_vBE\)-algebra. It is easy to verify that for any \(M \subseteq H\) that \(M \neq \{1\}\) and \(M \neq \{a\}\), \(M\) is a \(\delta\)-part.

**Corollary 4.11.** Let \((H, \circ, 1)\) be a \(H_vBE\)-algebra and \(M, N\) are \(\delta\)-part of \(H\). Then \(M \cap N\) is a \(\delta\)-part of \(H\).

**Proof:** For any \(n \in \mathbb{N}\), \(a_i \in H\), if \(L(a_1, a_2, \ldots, a_n) \cap (M \cap N) \neq \emptyset\), then \(L(a_1, a_2, \ldots, a_n) \cap M \neq \emptyset, L(a_1, a_2, \ldots, a_n) \cap N \neq \emptyset\). Since \(M, N\) are \(\delta\)-part, we have \(L(a_1, a_2, \ldots, a_n) \subseteq M, L(a_1, a_2, \ldots, a_n) \subseteq N\) and then \(L(a_1, a_2, \ldots, a_n) \subseteq M \cap N\). Therefore \(M \cap N\) is a \(\delta\)-part of \(H\).
Fundamental relation on $H_vBE$-Algebras

**Lemma 4.12** ([6]). Let $M$ be a non-empty subset of a dual hyper $K$-algebra $H$. Then the following conditions are equivalent:

(i) $M$ is a $\delta$-part of $H$,
(ii) $x \in M$, $x \delta y$ imply $y \in M$,
(iii) $x \in M$, $x \delta^* y$ imply $y \in M$.

**Theorem 4.13.** Let $(H, \circ, 1)$ be a $H_vBE$-algebra. If $H$ be a dual hyper $K$-algebra and for any $x \in H$, $\delta^*(x)$ is a $\delta$-part, then $\delta$ is transitive relation.

**Proof:** Let $x \delta y$ and $y \delta z$. Then there exist $m, n \in \mathbb{N}$, $a_i, b_j \in H$ such that $\{x, y\} \subseteq \bigcirc_{i=1}^{n} a_i$ and $\{y, z\} \subseteq \bigcirc_{j=1}^{m} b_j$. Now, $\delta^*(x)$ is a $\delta$-part, $x \in \delta^*(x) \cap \bigcirc_{i=1}^{n} a_i$ and $y \in \delta^*(x) \cap \bigcirc_{j=1}^{m} b_j$. Since $\delta^*(x)$ is a $\delta$-part, then $\bigcirc_{i=1}^{n} a_i \subseteq \delta^*(x)$ therefore $y \in \delta^*(x) \cap \bigcirc_{j=1}^{m} b_j$. Since $\delta^*(x)$ is a $\delta$-part, then $\bigcirc_{j=1}^{m} b_j \subseteq \delta^*(x)$ therefore $z \in \delta^*(x)$. But $z \in \delta(z)$ by above $z \delta^* x$, set $M = \delta(z)$ and know that $\delta^*(x) = \delta(z)$ then by Lemma 4.12, $x \delta z$, therefore $\delta$ is transitive relation.

**Open problem:** Under what conditions converse of above theorem is true?

5. Conclusion

In the present paper, we have introduced new $H_vBE$-algebras and $BE$-algebras based on equivalence relations.

This work focused on fundamental relations on $H_vBE$-algebras and we investigated some of their properties. The relations $\delta^*$ and $\delta$ are constructed and studied, they are one of the most main tools for better understanding the algebraic hyperstructures. In future, we try to find an answer to above open problem.
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References


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