Bulletin of the **S**ection of **L**ogic
Volume 51/2 (2022), pp. 243–266
https://doi.org/10.18778/0138-0680.2022.12

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A CLASSIFICATION OF IMPROPER INFERENCE RULES

Abstract

In the natural deduction system for classical propositional logic given by G. Gentzen, there are some inference rules with assumptions discharged by the rule. D. Prawitz calls such inference rules improper as opposed to proper ones. Improper inference rules are more complicated than proper ones and more difficult to understand. In 2022, we provided a sequent system based solely on the application of proper rules. In the present paper, on the basis of our system from 2022, we classify improper inference rules.

Keywords: Sequent system, improper inference rule, natural deduction.

2020 Mathematical Subject Classification: 03B05.

1. Introduction

In the natural deduction system for the classical propositional logic given in Gentzen [4], there are some inference rules with assumptions discharged by the rule. For instance, the implication introduction rule and the disjunction elimination rule have such assumptions. Prawitz [7] calls such inference rules improper as opposed to proper ones. The difference occurring between these two kinds of rules has been acknowledged by Fine [3], Robering [8], and Breckenridge and Magidor [1]. Nevertheless, no formal system allows for distinguishing them.

In Sasaki [9], we provided a sequent system $\vdash_{\mathbf{Sc}}$ that admits only proper rules. In the present paper, we classify the improper inference rules using

Presented by: Andrzej Indrzejczak
Received: October 24, 2021
Published online: June 23, 2022

$\vdash_{\mathbf{Sc}}$, however, there are still some that do not fall within our framework. The range of the inference rules considered is determined in Subsection 1.2, while the preliminaries are provided in Subsection 1.1, and our objectives are made specific in Subsection 1.3.

1.1. Preliminaries

Formulas are constructed from $\perp$ (contradiction) and propositional variables by means of logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), and $\rightarrow$ (implication) in the usual way. Latin letters p, q, and r, with or without subscripts, stand for propositional variables, and Greek letters ϕ, ψ, and χ, with or without subscripts, for formulas. The set of formulas is denoted by **Wff**. We define $\neg\varphi$ as $\varphi \rightarrow \perp$. Also, we use U and V, with or without subscripts, for sets of formulas, especially we use Greek letters $\Gamma, \Delta, \cdots$, with or without subscripts, for finite sets of formulas.

A *sequent* is an expression $(\Gamma \Rightarrow \varphi)$. We often write

$$\varphi_1, \cdots, \varphi_i, \Gamma_1, \cdots, \Gamma_j \Rightarrow \varphi$$

instead of

$$(\{\varphi_1, \cdots, \varphi_i\} \cup \Gamma_1 \cup \cdots \cup \Gamma_j \Rightarrow \varphi).$$

Sequents are denoted by X, Y, and Z, with or without subscripts. The *antecedent* $\mathbf{ant}(\Gamma \Rightarrow \varphi)$ and the *succedent* $\mathbf{suc}(\Gamma \Rightarrow \varphi)$ of a sequent $\Gamma \Rightarrow \varphi$ are defined as

$$\mathbf{ant}(\Gamma \Rightarrow \varphi) = \Gamma \quad \text{and} \quad \mathbf{suc}(\Gamma \Rightarrow \varphi) = \varphi,$$

respectively. We use S and T, with or without subscripts, for sets of sequents.

A *sequent system* is defined as a collection comprising a set **Axi** of sequents and a set **Inf** of inference rules of the form

$$\frac{X_1 \quad \cdots \quad X_n}{X} I.$$

Specifically, a *proof figure* of X from T in the sequent system is defined by means of the set $\mathbf{Axi} \cup T$ as axioms and **Inf** as inference rules in the usual way. Let $\vdash$, with or without subscripts, represent sequent systems and write $T \vdash X$ if there exists a proof figure of X from T in $\vdash$. We write $\vdash X$

and $T, U \vdash X$ instead of $\emptyset \vdash X$ and $T \cup \{\Rightarrow \phi \mid \phi \in U\} \vdash X$, respectively. Also, we write $T \vdash \Gamma \Rightarrow \Delta$ if $T \vdash \Gamma \Rightarrow \psi$ for every $\psi \in \Delta$. We note

$$T \nvdash \Gamma \Rightarrow \Delta \iff T \nvdash \Gamma \Rightarrow \psi \text{ for some } \psi \in \Delta.$$

We often take the recourse to the following properties:

- $T_1 \vdash X$ implies $T_1 \cup T_2 \vdash X$,
- $T_1 \vdash X$ and $T_2 \cup \{X\} \vdash Y$ imply $T_1 \cup T_2 \vdash Y$,

which can be shown by induction on a proof figure.

For a sequent system for the classical propositional logic, we use the system $\vdash_{\mathbf{Gc}}$ which corresponds to the natural deduction system in Gentzen [4] and Prawitz [7]. Specifically, the system $\vdash_{\mathbf{Gc}}$ is defined as follows.

Definition 1.1. A proof figure of X from T in $\vdash_{\mathbf{Gc}}$ is defined by means of the following axioms and inference rules.
Axioms:

- $\phi \Rightarrow \phi$,
- $\bot \Rightarrow \phi$,
- members of T.

Inference rules: See Figure 1.

We note that there are three improper inference rules ($\vee \Rightarrow$), ($\Rightarrow \rightarrow$), and (RAA) in Figure 1.

A sequent system $\vdash_{\mathbf{Sc}}$ is defined as follows.

Definition 1.2. A proof figure of X from T in the system $\vdash_{\mathbf{Sc}}$ is defined by means of the following axioms and inference rules.
Axioms: members of $\{X \mid \vdash_{\mathbf{Gc}} X\} \cup T$,
Inference rules: ($w \Rightarrow$) and (cut).

The system $\vdash_{\mathbf{Sc}}$ has only structural rules, and all logical content is put into axiomatic sequents. Such systems have been considered in Hertz [5], Suszko [11], Suszko [12], and Schroeder-Heister [10]. Indrzejczak [6] provides a brief summary of these works. However, a difference between proper and improper inference rules is not discussed there. [9] proved that $\vdash_{\mathbf{Sc}}$ distinguishes them, specifically,

$$\frac{\Gamma \Rightarrow \psi}{\phi, \Gamma \Rightarrow \psi}(w \Rightarrow) \qquad \frac{\Gamma \Rightarrow \phi \quad \phi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \psi}(\mathrm{cut})$$

$$\frac{\phi_1, \phi_2, \Gamma \Rightarrow \psi}{\phi_1 \wedge \phi_2, \Gamma \Rightarrow \psi}(\wedge \Rightarrow) \qquad \frac{\Gamma \Rightarrow \phi_1 \qquad \Gamma \Rightarrow \phi_2}{\Gamma \Rightarrow \phi_1 \wedge \phi_2}(\Rightarrow \wedge)$$

$$\frac{\phi_1, \Gamma \Rightarrow \psi \qquad \phi_2, \Gamma \Rightarrow \psi}{\phi_1 \vee \phi_2, \Gamma \Rightarrow \psi}(\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \phi_i}{\Gamma \Rightarrow \phi_1 \vee \phi_2}(\Rightarrow \vee)(i = 1, 2)$$

$$\frac{\Gamma \Rightarrow \phi_1 \qquad \phi_2, \Gamma \Rightarrow \psi}{\phi_1 \to \phi_2, \Gamma \Rightarrow \psi}(\to\Rightarrow) \qquad \frac{\phi_1, \Gamma \Rightarrow \phi_2}{\Gamma \Rightarrow \phi_1 \to \phi_2}(\Rightarrow\to)$$

$$\frac{\neg\phi, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow \phi}(\mathrm{RAA})$$

Figure 1. Inference rules of $\vdash_{\mathbf{Gc}}$

LEMMA 1.3. *Among the inference rules in Figure 1, none of the three improper inference rules* $(\vee \Rightarrow)$, $(\Rightarrow\to)$, *and* (RAA) *holds in* $\vdash_{\mathbf{Sc}}$, *and the other proper ones hold in* $\vdash_{\mathbf{Sc}}$.

To classify inference rules, we use the following system.

DEFINITION 1.4. For an inference rule I, we define $\vdash_{\mathbf{Sc}+I}$ as the system obtained by adding I to $\vdash_{\mathbf{Sc}}$.

The definition of the usual truth valuation is as follows.

DEFINITION 1.5. We say that a mapping $v : \mathbf{Wff} \to \{\mathsf{t}, \mathsf{f}\}$ is a *truth valuation* if the following conditions hold:

- $v(\bot) = \mathsf{f}$,
- $v(\phi \wedge \psi) = \mathsf{t} \iff v(\phi) = v(\psi) = \mathsf{t}$,
- $v(\phi \vee \psi) = \mathsf{f} \iff v(\phi) = v(\psi) = \mathsf{f}$,
- $v(\phi \to \psi) = \mathsf{f} \iff v(\phi) = \mathsf{t}$ and $v(\psi) = \mathsf{f}$.

We use u and v, with or without subscripts, for truth valuations. We write $v(U) = \mathsf{t}$ if $v(\phi) = \mathsf{t}$ for each $\phi \in U$. The completeness below can be shown in the usual way. For example, we can refer to Chagrov and Zakharyaschev [2].

LEMMA 1.6. *The following conditions are equivalent:*

(1) $U \vdash_{\mathbf{Gc}} \Rightarrow \phi$,

(2) *for any* v, $v(U) = \mathsf{t}$ *implies* $v(\phi) = \mathsf{t}$.

We modify the above truth valuation as follows.

DEFINITION 1.7. Let $\mathbf{v}$ be a set of truth valuations. We define a mapping $\mathbf{v} : \mathbf{Wff} \to \{\mathsf{t}, \mathsf{f}\}$ as follows:

$$\mathbf{v}(\phi) = \mathsf{t} \iff \text{for any } v \in \mathbf{v},\ v(\phi) = \mathsf{t}.$$

We note that

- $\emptyset(\phi) = \mathsf{t}$,
- $\{v\}(\phi) = v(\phi)$,
- $\{v_1, v_2\}(\phi) = \mathsf{t} \iff v_1(\phi) = v_2(\phi) = \mathsf{t}$.

We use $\mathbf{u}$ and $\mathbf{v}$, with or without subscripts, for a set of truth valuations. We write $\mathbf{v}(U) = \mathsf{t}$ if $\mathbf{v}(\phi) = \mathsf{t}$ for each $\phi \in U$. Also, we write $\mathbf{v}(X) = \mathsf{t}$ if $\mathbf{v}(\mathbf{ant}(X)) = \mathsf{f}$ or $\mathbf{v}(\mathbf{suc}(X)) = \mathsf{t}$. Moreover, we write $\mathbf{v}(T) = \mathsf{t}$ if $\mathbf{v}(X) = \mathsf{t}$ for each $X \in T$. We define $\#(\mathbf{v})$ as the number of elements in $\mathbf{v}$.

The following lemma was proved in [9].

LEMMA 1.8. *The following conditions are equivalent:*

(1) $T \vdash_{\mathbf{Sc}} X$,

(2) *for any* $\mathbf{v}$, $\mathbf{v}(T) = \mathsf{t}$ *implies* $\mathbf{v}(X) = \mathsf{t}$.

1.2. The range of inference rules

In the present paper, we classify the improper inference rules. However, there are some inference rules which are not considered here. Also, the meaning of "improper" has not been clear. Prawitz [7] calls an inference rule in the natural deduction system improper if it has some assumptions discharged by the rule. For example, the rules in natural deduction system corresponding to $(\vee \Rightarrow)$, $(\Rightarrow\to)$, and (RAA) are improper and the rules corresponding to $(\wedge \Rightarrow)$, $(\Rightarrow \wedge)$, $(\Rightarrow \vee)$, and $(\rightarrow\Rightarrow)$ are proper. However, there are some rules in natural deduction systems we can not distinguish between proper or improper. In the present subsection, we give our interpretation of improper rules in sequent systems and clarify the range of the inference rules we investigate here.

First, we give our interpretation of "improper inference rule". [9] gave a condition (C2), and confirmed that (C2) is a natural interpretation of "improper" among inference rules which hold in $\vdash_{\mathbf{Gc}}$ by showing the equivalence between $T \nvdash_{\mathbf{Sc}} X$ and (C2). Here, we use this interpretation. Also, if an inference rule I does not hold in $\vdash_{\mathbf{Gc}}$, then we treat I as an improper one. Specifically, an inference rule

$$\frac{X_1 \quad \cdots \quad X_m}{X}$$

is improper if the following condition (C2) holds:

(C2) there exists a subset T of $\{X_1, \cdots, X_m\}$ satisfying the following two conditions:

- $\{X_1, \cdots, X_m\} \backslash T \nvdash_{\mathbf{Gc}} \mathbf{ant}(X) \Rightarrow \mathbf{ant}(X_i)$ for each $X_i \in T$,
- $\{X_1, \cdots, X_m\} \backslash T \nvdash_{\mathbf{Gc}} X$.

Since (C2) is equivalent to $T \nvdash_{\mathbf{Sc}} X$, an improper inference rule is a rule which does not hold in $\vdash_{\mathbf{Sc}}$.

Below, we also clarify a range of the improper inference rule we investigate. Such inference rules are the following adequate ones. It will be shown in Lemma 1.10 that an inference rule I is adequate if $\vdash_{\mathbf{Sc}+I}$ is complete for some class of sets for truth valuations. From this point of view, we treat most of natural inference rules.

Definition 1.9. We say that an inference rule I is *adequate* if the following two conditions hold:

(1)
$$\{\Gamma_i \Rightarrow \phi_i \mid i \in \{1, \cdots, m\}\} \vdash_{\mathbf{Sc}+I} \Gamma_0 \Rightarrow \phi_0$$

implies

$$\{\Gamma, \Gamma_i \Rightarrow \phi_i \mid i \in \{1, \cdots, m\}\} \vdash_{\mathbf{Sc}+I} (\Gamma, \Gamma_0 \Rightarrow \phi_0),$$

(2)

$$T \cup \{\Gamma_1 \Rightarrow \phi_1\} \vdash_{\mathbf{Sc}+I} \Gamma_0 \Rightarrow \phi_0$$

implies

$$T \cup \{\Gamma_1 \Rightarrow \phi_0\} \vdash_{\mathbf{Sc}+I} \Gamma_0 \Rightarrow \phi_0.$$

LEMMA 1.10. *Let I be an inference rule. If there exists a class $\mathcal{C}$ of sets of truth valuations satisfying the completeness:*

$$T \vdash_{\mathbf{Sc}+I} X \iff \mathbf{v}(T) = \mathsf{t} \textit{ implies } \mathbf{v}(X) = \mathsf{t}, \textit{ for every } \mathbf{v} \in \mathcal{C},$$

then I is adequate.

PROOF: Here, we also use (1) and (2) as the conditions (1) and (2) in Definition 1.9, respectively. We can show (1), by the existence of $\mathcal{C}$.

We show (2). Suppose that

$$T \cup \{\Gamma_1 \Rightarrow \phi_1\} \vdash_{\mathbf{Sc}+I} \Gamma_0 \Rightarrow \phi_0$$

and let $\mathbf{v}$ be a set in $\mathcal{C}$. Then by the completeness, we have

$$\mathbf{v}(T) = \mathbf{v}(\Gamma_0) = \mathbf{v}(\Gamma_1 \Rightarrow \phi_1) = \mathsf{t} \text{ implies } \mathbf{v}(\phi_0) = \mathsf{t}. \tag{3}$$

If

$$\mathbf{v}(T) = \mathbf{v}(\Gamma_0) = \mathsf{t} \text{ implies } \mathbf{v}(\Gamma_1) = \mathsf{t}, \tag{4}$$

then we have

$$\mathbf{v}(T) = \mathbf{v}(\Gamma_0) = \mathbf{v}(\Gamma_1 \Rightarrow \phi_0) = \mathsf{t} \text{ implies } \mathbf{v}(\phi_0) = \mathsf{t}. \tag{5}$$

If (4) does not hold, then we have

$$\mathbf{v}(T) = \mathbf{v}(\Gamma_0) = \mathsf{t} \text{ and } \mathbf{v}(\Gamma_1) = \mathsf{f},$$

and so,

$$\mathbf{v}(T) = \mathbf{v}(\Gamma_0) = \mathbf{v}(\Gamma_1 \Rightarrow \phi_1) = \mathsf{t},$$

and using (3), we have $\mathbf{v}(\phi_0) = \mathsf{t}$. So, in both cases, we have (5). Using the completeness, we obtain

$$T \cup \{\Gamma_1 \Rightarrow \phi_0\} \vdash_{\mathbf{Sc}+I} \Gamma_0 \Rightarrow \phi_0. \qquad \square$$

1.3. The purposes

In the present paper, we classify the adequate improper inference rules. In other words, we classify the systems obtained from $\vdash_{\mathbf{Sc}}$ by adding an adequate inference rule. In the present subsection, we describe this purpose precisely.

In the present paper, we fix the variable n for a natural number. We use $\vec{p}$ for the sequence $p_1, \cdots, p_n$. Similarly, we also use $\vec{\phi}$, $\vec{\psi}$, $\vec{\chi}$, and so on.

We consider the inference rule

$$\frac{\mathrm{ed}_{n,0}(\vec{\phi}), \Gamma \Rightarrow \psi \quad \cdots \quad \mathrm{ed}_{n,n}(\vec{\phi}), \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \psi} \ (\mathrm{em}_n),$$

where

$$\mathrm{ed}_{n,k}(\vec{\phi}) = \bigvee_{1 \le i \le k} \neg\phi_i \vee \bigvee_{k < j \le n} \phi_j \text{ for every } k \in \{0, \cdots, n\}.$$

We note that

$$\mathrm{ed}_{3,0}(\vec{\phi}) = \phi_1 \vee \phi_2 \vee \phi_3, \qquad \mathrm{ed}_{3,1}(\vec{\phi}) = \neg\phi_1 \vee \phi_2 \vee \phi_3,$$

$$\mathrm{ed}_{3,2}(\vec{\phi}) = \neg\phi_1 \vee \neg\phi_2 \vee \phi_3, \qquad \mathrm{ed}_{3,3}(\vec{\phi}) = \neg\phi_1 \vee \neg\phi_2 \vee \neg\phi_3.$$

We often call these formulas elementary disjunctions if each ϕ_i is a propositional variable. The name "ed" is intended to mean "elementary disjunction". We also note that (em_1) and (em_2) are as follows:

$$\frac{\phi_1, \Gamma \Rightarrow \psi \qquad \neg\phi_1, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \psi} \ (\mathrm{em}_1),$$

$$\frac{\phi_1 \vee \phi_2, \Gamma \Rightarrow \psi \quad \neg\phi_1 \vee \phi_2, \Gamma \Rightarrow \psi \quad \neg\phi_1 \vee \neg\phi_2, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \psi} \ (\mathrm{em}_2),$$

and we find that (em_1) corresponds to the law of excluded middle. The name "(em_n)" is intended to mean the law. We write $T \vdash_n X$ instead of $T \vdash_{\mathbf{Sc}+(\mathrm{em}_n)} X$.

The main theorem in the present paper is as follows.

Theorem 1.11.

(1) $T \vdash_{\mathbf{Gc}} X \iff T \vdash_1 X$.
(2) $T \vdash_{\mathbf{Sc}} X \Longrightarrow T \vdash_n X$.

(3) $T \vdash_{n+1} X \Longrightarrow T \vdash_n X$.
(4) *Neither of the converses of* (2) *and* (3) *holds.*
(5) *For each adequate inference rule* I*, either one of the following three conditions holds:*

(5.1) $\vdash_{\mathbf{Sc}+I} \Rightarrow \bot$,
(5.2) $T \vdash_{\mathbf{Sc}+I} X \iff T \vdash_n X$*, for some* n*,*
(5.3) $T \vdash_{\mathbf{Sc}+I} X \iff T \vdash_{\mathbf{Sc}} X$.

In the following section, we prove the completeness of $\vdash_n$ and show the first four conditions in Theorem 1.11. In Section 3, we prove Theorem 1.11(5).

2. The system $\vdash_n$

In the present section, we prove the completeness of $\vdash_n$ and show the first four conditions in Theorem 1.11.

THEOREM 2.1. *The following conditions are equivalent:*

(1) $T \vdash_n X$,
(2) *for each* $\mathbf{v}$, $\#(\mathbf{v}) \leq n$ *and* $\mathbf{v}(T) = \mathsf{t}$ *imply* $\mathbf{v}(X) = \mathsf{t}$.

In order to prove Theorem 2.1, we provide some preparations.

LEMMA 2.2.

(1) $T \cup \{\Rightarrow \psi\} \vdash_n \Gamma \Rightarrow \phi \iff T \vdash_n (\psi, \Gamma \Rightarrow \phi)$.
(2) $U \vdash_{\mathbf{Sc}} X \iff U \vdash_n X \iff U \vdash_{\mathbf{Gc}} X$.

PROOF: By induction on a proof figure. □

LEMMA 2.3. *If* $\#(\mathbf{v}) \leq n$*, then there exists* $k \in \{0, \cdots, n\}$ *such that* $\mathbf{v}(\mathrm{ed}_{n,k}(\vec{\phi})) = \mathsf{t}$.

PROOF: In the present proof, we write $\mathrm{ed}_{n,k}$ instead of $\mathrm{ed}_{n,k}(\vec{\phi})$. It is sufficient to show

$$\#(\mathbf{v}) \leq n \text{ and } \mathbf{v}(\mathrm{ed}_{n,1}) = \cdots = \mathbf{v}(\mathrm{ed}_{n,n}) = \mathsf{f} \text{ imply } \mathbf{v}(\mathrm{ed}_{n,0}) = \mathsf{t}.$$

Suppose that $\#(\mathbf{v}) \leq n$ and $\mathbf{v}(\mathrm{ed}_{n,1}) = \cdots = \mathbf{v}(\mathrm{ed}_{n,n}) = \mathsf{f}$. Then, for each $k \in \{1, \cdots, n\}$, there exists $v_k \in \mathbf{v}$ such that $v_k(\mathrm{ed}_{n,k}) = \mathsf{f}$, and so,

$$v_k(\phi_1) = \cdots = v_k(\phi_{k-1}) = \mathsf{t} \text{ and } v_k(\phi_k) = \cdots = v_k(\phi_n) = \mathsf{f}.$$

Therefore, for each $i, j \in \{1, \cdots, n\}$ with $i < j$, we observe $v_i(\mathrm{ed}_{n,i}) = \mathsf{f}$ and $v_j(\mathrm{ed}_{n,i}) = \mathsf{t}$, and so, we have $v_i \neq v_j$. Therefore, we have $\#(\{v_1, \cdots, v_n\}) = n$. Using $\{v_1, \cdots, v_n\} \subseteq \mathbf{v}$ and $\#(\mathbf{v}) \leq n$, we have $\{v_1, \cdots, v_n\} = \mathbf{v}$. We also observe that

$$v_k(\phi_1) = \mathsf{t} \text{ for every } v_k \in \mathbf{v},$$

and so,

$$v_k(\mathrm{ed}_{n,0}) = \mathsf{t} \text{ for every } v_k \in \mathbf{v}.$$

Hence, we obtain $\mathbf{v}(\mathrm{ed}_{n,0}) = \mathsf{t}$. □

Corollary 2.4. If $\#(\mathbf{v}) \leq n$ and

$$\mathbf{v}(\mathrm{ed}_{n,k}(\vec{\phi}), \Gamma \Rightarrow \phi) = \mathsf{t}, \text{ for every } k \in \{0, \cdots, n\},$$

then $\mathbf{v}(\Gamma \Rightarrow \phi) = \mathsf{t}$.

Proof: By Lemma 2.3. □

Lemma 2.5. *If $T \vdash_n X$, then for each $\mathbf{v}$,*

$$\#(\mathbf{v}) \leq n \textit{ and } \mathbf{v}(T) = \mathsf{t} \textit{ imply } \mathbf{v}(X) = \mathsf{t}.$$

Proof: We use induction on a proof figure of $T \vdash_n X$.

Basis. If $X \in T$, then the lemma is clear. If $\vdash_{\mathbf{Gc}} X$, then we obtain the lemma by means of Lemma 1.6.

Induction step can be shown by Corollary 2.4 and the following two:

- $\#(\mathbf{v}) \leq n$ and $\mathbf{v}(\Gamma \Rightarrow \psi) = \mathsf{t}$ imply $\mathbf{v}(\phi, \Gamma \Rightarrow \psi) = \mathsf{t}$,
- $\#(\mathbf{v}) \leq n$ and $\mathbf{v}(\Gamma \Rightarrow \phi) = \mathbf{v}(\phi, \Gamma \Rightarrow \psi) = \mathsf{t}$ imply $\mathbf{v}(\Gamma \Rightarrow \psi) = \mathsf{t}$. □

Lemma 2.6. *If $T \not\vdash_n \Gamma \Rightarrow \phi$, then there exists a pair $\langle U, V \rangle$ satisfying the following four conditions:*

(1) *$\Gamma \subseteq U$ and $\phi \in V$,*

(2) *$U \cup V = \mathbf{Wff}$,*

(3) *$T, U \not\vdash_n \Rightarrow \psi$ for each $\psi \in V$,*

(4) *for each $\vec{\phi}$, $\mathbf{ed}_n \not\subseteq V$, where*

$$\mathbf{ed}_n = \{\mathrm{ed}_{n,k}(\vec{\phi}) \mid k \in \{0, \cdots, n\}\}.$$

PROOF: Suppose that $T \not\vdash_n \Gamma \Rightarrow \phi$. To define $\langle U, V\rangle$ satisfying the required conditions, we construct a list $U_1, U_2, \cdots$. Let

$$\chi_1^*, \chi_2^*, \cdots$$

be an enumeration of all formulas. We consider the list

$$\chi_1^*, \chi_2^*, \chi_1^*, \chi_2^*, \chi_3^*, \chi_1^*, \chi_2^*, \chi_3^*, \chi_4^*, \cdots ,$$

and we write χ_m instead of the m-th formula of the list above. We note that for each integer i and each formula χ, there exists an integer j such that $\chi = \chi_j$ and $i < j$. We define U_k as

- $U_0 = \Gamma$,
- $U_{k+1} = \begin{cases} U_k \cup \{\chi_{k+1}\} & \text{if } T \not\vdash_n (\chi_{k+1}, U_k \Rightarrow \phi) \\ U_k & \text{otherwise.} \end{cases}$

We note that

$$i < j \text{ implies } U_i \subseteq U_j. \tag{0.1}$$

We define the pair $\langle U, V\rangle$ as

$$U = \bigcup_{i=0}^{\infty} U_i \quad \text{ and } V = \mathbf{Wff} \backslash U.$$

We show the required conditions (1), (2), (3), and (4). (2) is clear from the definition.

For (1). $\Gamma \subseteq U$ is clear from the definition. Suppose that $\phi \notin V$. Then by (2), we have $\phi \in U$, and so

$$\phi \in U_k \text{ for some } k. \tag{2.1}$$

On the other hand, by induction on k, we can show

$$T \not\vdash_n U_k \Rightarrow \phi \text{ for every } k, \tag{2.2}$$

which is in contradiction with (2.1).

For (3). Suppose that $T, U \vdash_n \Rightarrow \psi$ for some $\psi \in V$. Then by Lemma 2.2(1), there exists k' such that

$$T \vdash_n U_{k'} \Rightarrow \psi. \tag{3.1}$$

By the definition of the list $\chi_1, \chi_2, \cdots$, we observe that $\psi = \chi_{j+1}$ for some $j \geq k'$. By $j \geq k'$ and (0.1), we have $U_{k'} \subseteq U_j$. Also, by $\psi = \chi_{j+1} \in V$, we have $T \vdash_n (\psi, U_j \Rightarrow \phi)$. Using (2.2) and (cut), we have $T \not\vdash_n U_j \Rightarrow \psi$, which is in contradiction with (3.1) and $U_{k'} \subseteq U_j$.

For (4). Suppose that $\mathbf{ed}_n \subseteq V$. Then for each $\mathrm{ed} \in \mathbf{ed}_n \subseteq V$, there exists an integer $f(\mathrm{ed})$ such that $\mathrm{ed} = \chi_{f(\mathrm{ed})+1}$ and

$$T \vdash_n (\mathrm{ed}, U_{f(\mathrm{ed})} \Rightarrow \phi).$$

Let m be the maximum in $\{f(\mathrm{ed}) \mid \mathrm{ed} \in \mathbf{ed}_n\}$. Then, using (0.1), we have

$$T \vdash_n (\mathrm{ed}, U_m \Rightarrow \phi) \text{ for every } \mathrm{ed} \in \mathbf{ed}_n,$$

and using (em_n), we have

$$T \vdash_n U_m \Rightarrow \phi,$$

which is in contradiction with (2.2). □

Lemma 2.7. *If* $\#(\{v_1, \cdots, v_{n+1}\}) = n+1$*, then there exist* $\phi_1, \cdots, \phi_n$ *such that*

$$v_i(\phi_j) = \mathsf{f} \iff i \leq j.$$

Proof: Let a and b be members in $\{1, \cdots, n+1\}$ with $a < b$. By $\#(\{v_1, \cdots, v_{n+1}\}) = n+1$, we have $v_a \neq v_b$, and so, there exists a propositional variable $p_{a,b}$ such that $v_a(p_{a,b}) \neq v_b(p_{a,b})$. We define $\psi_{a,b}$ as

$$\psi_{a,b} = \begin{cases} p_{a,b} & \text{if } v_a(p_{a,b}) = \mathsf{f} \\ \neg p_{a,b} & \text{if } v_a(p_{a,b}) = \mathsf{t}. \end{cases}$$

Then we observe

$$v_a(\psi_{a,b}) = \mathsf{f},$$

and using $v_a(p_{a,b}) \neq v_b(p_{a,b})$,

$$v_b(\psi_{a,b}) = \mathsf{t}.$$

For $\ell \in \{1, \cdots, n+1\}$, we define ψ_ℓ as

$$\psi_\ell = \bigvee_{\ell < k \leq n+1} \psi_{\ell,k}.$$

Table 1. A truth table of formulas in Lemma 2.7 in the case $n = 3$

	$\psi_{1,2}$	$\psi_{1,3}$	$\psi_{1,4}$	ψ_1	ϕ_1	$\psi_{2,3}$	$\psi_{2,4}$	ψ_2	ϕ_2	$\psi_{3,4}$	ψ_3	ϕ_3
v_1	f	f	f	f	f				f			f
v_2	t			t	t	f	f	f	f			f
v_3		t		t	t	t		t	t	f	f	f
v_4			t	t	t		t	t	t	t	t	t

Then we have

$$v_a(\psi_a) = \mathsf{f} \text{ and } v_{a+1}(\psi_a) = \cdots = v_{n+1}(\psi_a) = \mathsf{t}.$$

Finally, we define ϕ_i as

$$\phi_j = \bigwedge_{1 \le k \le j} \psi_k.$$

If $i \le j$, then by $v_i(\psi_i) = \mathsf{f}$, we have $v_i(\phi_j) = \mathsf{f}$. If $i > j$, then by $v_i(\psi_k) = \mathsf{t}$ for each $k \in \{1, \cdots, i-1\}$, we have $v_i(\phi_j) = \mathsf{t}$. $\square$

In Table 1, we can see the truth values of the formulas occurring in Lemma 2.7 in the case that $n = 3$.

Lemma 2.8. *If $T \not\vdash_n \Gamma \Rightarrow \phi$, then there exists* $\mathbf{v}$ *such that* $\#(\mathbf{v}) \le n$, $\mathbf{v}(T) = \mathsf{t}$, *and* $\mathbf{v}(\Gamma \Rightarrow \phi) = \mathsf{f}$.

Proof: Suppose that $T \not\vdash_n \Gamma \Rightarrow \phi$. By Lemma 2.6, there exists a pair $\langle U, V \rangle$ satisfying the following four conditions:

(1) $\Gamma \subseteq U$ and $\phi \in V$,

(2) $U \cup V = \mathbf{Wff}$,

(3) $T, U \not\vdash_n \Rightarrow \psi$ for each $\psi \in V$,

(4) for each $\vec{\phi}$, $\mathbf{ed}_n (= \{\mathrm{ed}_{n,k}(\vec{\phi}) \mid k \in \{0, \cdots, n\}\}) \not\subseteq V$.

By (3), for each $\psi \in V$, we observe

$$U \not\vdash_n \Rightarrow \psi,$$

and by Lemma 2.2, we have

$$U \not\vdash_{\mathbf{Gc}} \Rightarrow \psi,$$

and by Lemma 1.6, there exists v_ψ satisfying

$$v_\psi(U) = \mathsf{t} \text{ and } v_\psi(\psi) = \mathsf{f}.$$

We define $\mathbf{v}$ as

$$\mathbf{v} = \{v_\psi \mid \psi \in V\}.$$

Then we have

$$\mathbf{v}(U) = \mathsf{t} \text{ and } \mathbf{v}(\psi) = \mathsf{f} \text{ for each } \psi \in V,$$

and using (1), we have

$$\mathbf{v}(\Gamma \Rightarrow \phi) = \mathsf{f}.$$

Also, similarly to Lemma 2.10 in [9], we can show $\mathbf{v}(T) = \mathsf{t}$.

So, we have only to show $\#(\mathbf{v}) \leq n$. Suppose that $\#(\mathbf{v}) \geq n+1$. Then there exist $\psi_1, \cdots, \psi_{n+1} \in V$ such that $\#(\{v_{\psi_1}, \cdots, v_{\psi_{n+1}}\}) = n+1$. By Lemma 2.7, there exist $\chi_1, \cdots, \chi_n$ such that

$$v_{\psi_i}(\chi_j) = \mathsf{f} \iff i \leq j,$$

and by the definition of $\mathrm{ed}_{n,i}(\vec{\chi})$,

$$v_{\psi_i}(\mathrm{ed}_{n,i}(\vec{\chi})) = \mathsf{f},$$

and using $v_{\psi_i} \in \mathbf{v}$,

$$\mathbf{v}(\mathrm{ed}_{n,i}(\vec{\chi})) = \mathsf{f},$$

Using $\mathbf{v}(U) = \mathsf{t}$ and (2), we have $\mathbf{ed}_n \subseteq V$, which is in contradiction with (4). $\square$

From Lemma 2.5 and Lemma 2.8, we obtain Theorem 2.1. The first three conditions in Theorem 1.11 are obtained by Theorem 2.1, Lemma 1.6, and Lemma 1.8. Theorem 1.11(4) is obtained by the following lemma.

Lemma 2.9. *The rule* (em_n) *does not hold in* $\vdash_{n+1}$.

Proof: We show that the following rule does not hold in $\vdash_{n+1}$:

$$\frac{\mathrm{ed}_{n,0}(\vec{p}) \Rightarrow \bot \quad \cdots \quad \mathrm{ed}_{n,n}(\vec{p}) \Rightarrow \bot}{\Rightarrow \bot}.$$

Table 2. A truth table of formulas in Lemma 2.9 in the case $n = 3$

	p_1	p_2	p_3	$\mathrm{ed}_{3,0}(\vec{p})$	$\mathrm{ed}_{3,1}(\vec{p})$	$\mathrm{ed}_{3,2}(\vec{p})$	$\mathrm{ed}_{3,3}(\vec{p})$
v_0	f	f	f	f	t	t	t
v_1	t	f	f	t	f	t	t
v_2	t	t	f	t	t	f	t
v_3	t	t	t	t	t	t	f

By Lemma 2.5, we have only to show that there exists $\mathbf{v}$ such that

$$\#(\mathbf{v}) = n + 1 \text{ and } \mathbf{v}(\mathrm{ed}_{n,0}(\vec{p})) = \cdots = \mathbf{v}(\mathrm{ed}_{n,n}(\vec{p})) = \mathsf{f}. \tag{1}$$

We define $v_0, \cdots, v_n$ as

$$v_i(p_j) = \mathsf{f} \iff i < j$$

and define $\mathbf{v}$ as $\mathbf{v} = \{v_0, \cdots, v_n\}$. Then we observe

$$v_i(\mathrm{ed}_{n,j}(\vec{p})) = \mathsf{f} \iff i = j \text{ for every } i, j \in \{0, \cdots, n\}.$$

Therefore, we have $\#(\mathbf{v}) = n + 1$ and

$$v_k(\mathrm{ed}_{n,k}(\vec{p})) = \mathsf{f} \text{ for every } k \in \{0, \cdots, n\}.$$

Hence, we obtain (1). □

In Table 2, we can see the truth values of the formulas occurring in Lemma 2.9 in the case that $n = 3$.

3. A classification

In the present section, we prove Theorem 3.1. Theorem 1.11(5) will be obtained as a corollary of the theorem.

First, we define $\mathbf{v}(I)$ for an inference rule I. For an inference rule,

$$\frac{X_1 \quad \cdots \quad X_m}{X} I,$$

we write $\mathbf{v}(I) = \mathsf{t}$ if $\mathbf{v}(X_1) = \cdots = \mathbf{v}(X_m) = \mathsf{t}$ implies $\mathbf{v}(X) = \mathsf{t}$.

Theorem 3.1. *Let I be an adequate inference rule and let N be the set* $\{\#(\mathbf{v}) \mid \mathbf{v}(I) = \mathsf{f}\}$. *Then*

(1) $N \neq \emptyset$ *and* $\min N = 1$ *imply* $\vdash_{\mathbf{Sc}+I} \Rightarrow \bot$,

(2) $N \neq \emptyset$ *and* $\min N > 1$ *imply* $T \vdash_{\mathbf{Sc}+I} X \iff T \vdash_{\min N-1} X$,

(3) $N = \emptyset$ *implies* $T \vdash_{\mathbf{Sc}+I} X \iff T \vdash_{\mathbf{Sc}} X$.

(1) and (3) in the above theorem are obtained by Lemma 1.6 and Lemma 1.8, respectively. Below, we provide some preparations for Theorem 3.1(2).

We define the set $\mathbf{ED}_n(\vec{\phi})$ as

$$\mathbf{ED}_n(\vec{\phi}) = \{\bigvee_{i\in N} \neg\phi_i \vee \bigvee_{i\in\{1,\cdots,n\}\setminus N} \phi_i \mid N \subseteq \{1,\cdots,n\}\}$$

and consider the following inference rule:

$$\frac{\mathrm{ed}_{n,0}, \Gamma \Rightarrow \psi \ \cdots \ \mathrm{ed}_{n,n}, \Gamma \Rightarrow \psi \quad \Gamma \Rightarrow \mathrm{ed}_{n,n+1} \ \cdots \ \Gamma \Rightarrow \mathrm{ed}_{n,2^n-1}}{\Gamma \Rightarrow \psi}\ (\mathrm{em}_n^*),$$

where

$$\mathrm{ed}_{n,k} = \mathrm{ed}_{n,k}(\vec{\phi}) \text{ for every } k \in \{0,\cdots,n\}$$

and

$$\{\mathrm{ed}_{n,k}(\vec{\phi}) \mid k \in \{n+1,\cdots,2^n-1\}\} = \mathbf{ED}_n(\vec{\phi})\setminus\{\mathrm{ed}_{n,k}(\vec{\phi}) \mid 0 \leq k \leq n\}.$$

We also consider the rule (em_n^0) obtained from (em_n^*) by replacing $\vec{\phi}$ with $\vec{p}$, respectively. Moreover, we consider the rule $(\mathrm{em}_n^0(\bot))$ obtained from (em_n^0) by replacing ψ with $\bot$.

Remark 3.2.

$$\{\mathrm{ed}_{n,k}(\vec{\phi}) \mid 0 \leq k \leq n\}$$
$$= \left\{ \bigvee_{i\in N} \neg\phi_i \vee \bigvee_{i\in\{1,\cdots,n\}\setminus N} \phi_i \;\middle|\; N \in \{\emptyset, \{1\}, \cdots, \{1,\cdots,n\}\} \right\}.$$

Lemma 3.3.

$$T \vdash_{\mathbf{Sc}+(\mathrm{em}_n^*)} X \iff T \vdash_n X.$$

Proof: It is easily observed that (em_n^*) holds in $\vdash_n$.

We show that (em_n) holds in $\vdash_{\mathbf{Sc}+(\mathrm{em}^*_n)}$. Suppose that

$$T \vdash_{\mathbf{Sc}+(\mathrm{em}^*_n)} (\mathrm{ed}_{n,k}(\vec{\phi}), \Gamma \Rightarrow \psi) \text{ for every } k \in \{0, \cdots, n\}. \tag{1}$$

For each $k \in \{1, \cdots, n\}$, we define ψ_k as

$$\psi_k = \bigvee_{k \leq i \leq n} \phi_i.$$

Then $T \vdash_{\mathbf{Sc}+(\mathrm{em}^*_n)} (\Gamma \Rightarrow \psi)$ follows from

$$T \vdash_{\mathbf{Sc}+(\mathrm{em}^*_n)} (\mathrm{ed}_{n,k}(\vec{\psi}), \Gamma \Rightarrow \psi) \text{ for every } k \in \{0, \cdots, n\} \tag{2}$$

and

$$T \vdash_{\mathbf{Sc}+(\mathrm{em}^*_n)} \Gamma \Rightarrow \mathrm{ed} \text{ for every } \mathrm{ed} \in \mathbf{ED}_n(\vec{\psi}) \backslash \{\mathrm{ed}_{n,k}(\vec{\psi}) \mid 0 \leq k \leq n\}, \tag{3}$$

by (em^*_n). So, it is sufficient to show (2) and (3).

For (2). For each $k \in \{1, \cdots, n\}$, we note

$$\begin{array}{ll} & \vdash_{\mathbf{Sc}} \Rightarrow \mathrm{ed}_{n,k}(\vec{\psi}) \\ \iff & \vdash_{\mathbf{Sc}} \Rightarrow \bigvee_{1 \leq i \leq k} \neg\psi_i \vee \bigvee_{k < i \leq n} \psi_i \\ \iff & \vdash_{\mathbf{Sc}} \Rightarrow \bigvee_{1 \leq i \leq k} \neg\psi_i \vee \bigvee_{k < i \leq n} \phi_i \\ \iff & \vdash_{\mathbf{Sc}} \Rightarrow \bigvee_{1 \leq i < k} \neg\psi_i \vee \neg\psi_k \vee \bigvee_{k < i \leq n} \phi_i \\ \iff & \vdash_{\mathbf{Sc}} \Rightarrow \bigvee_{1 \leq i < k} \neg\psi_i \vee \neg(\phi_k \vee \bigvee_{k < i \leq n} \phi_i) \vee \bigvee_{k < i \leq n} \phi_i \\ \iff & \vdash_{\mathbf{Sc}} \Rightarrow \bigvee_{1 \leq i < k} \neg\psi_i \vee \neg\phi_k \vee \bigvee_{k < i \leq n} \phi_i \\ \iff & \vdash_{\mathbf{Sc}} \Rightarrow \neg\phi_k \vee \bigvee_{k < i \leq n} \phi_i \end{array}$$

and

$$\vdash_{\mathbf{Sc}} \Rightarrow \mathrm{ed}_{n,0}(\vec{\psi}) \iff \vdash_{\mathbf{Sc}} \Rightarrow \bigvee_{1 \leq i \leq n} \psi_i \iff \vdash_{\mathbf{Sc}} \Rightarrow \bigvee_{1 \leq i \leq n} \phi_i.$$

Therefore, for each $k \in \{0, \cdots, n\}$, we have

$$\vdash_{\mathbf{Sc}} \mathrm{ed}_{n,k}(\vec{\psi}) \Rightarrow \mathrm{ed}_{n,k}(\vec{\phi}),$$

and using (1), we obtain (2).

For (3). Suppose that $\mathrm{ed} \in \mathbf{ED}_n(\vec{\psi}) \backslash \{\mathrm{ed}_{n,k}(\vec{\psi}) \mid 0 \leq k \leq n\}$. Then by Remark 3.2, there exists a subset N of $\{1, \cdots, n\}$ such that

$$N \notin \{\emptyset, \{1\}, \{1,2\}, \cdots, \{1, \cdots, n\}\} \tag{4}$$

and

$$\mathrm{ed} = \bigvee_{i \in N} \neg\psi_i \vee \bigvee_{i \in \{1,\cdots,n\} \setminus N} \psi_i. \tag{5}$$

Then by (4), there exists $k \in \{2, \cdots, n\}$ such that

$$k \in N \text{ and } k-1 \notin N,$$

and using (5), we have

$$\vdash_{\mathbf{Sc}} \neg\psi_k \vee \psi_{k-1} \Rightarrow \mathrm{ed},$$

and by the definition of ψ_k, we have

$$\vdash_{\mathbf{Sc}} \neg\psi_k \vee (\phi_{k-1} \vee \psi_k) \Rightarrow \mathrm{ed},$$

Using $\vdash_{\mathbf{Sc}} \Rightarrow \neg\psi_k \vee (\phi_{k-1} \vee \psi_k)$, we obtain (3). □

LEMMA 3.4. *Let I be an inference rule. If there exists $\mathbf{u}$ such that $\#(\mathbf{u}) = n+1$ and $\mathbf{u}(I) = \mathsf{f}$, then there exists a substitution σ such that, for each $\mathbf{v}$,*

$$\mathbf{v}(\mathrm{em}_n^0(\bot)) = \mathsf{f} \textit{ implies } \mathbf{v}(I\sigma) = \mathsf{f}.$$

PROOF: Suppose that $\#(\mathbf{u}) = n+1$, $\mathbf{u}(I) = \mathsf{f}$, and $\mathbf{u} = \{u_0, \cdots, u_n\}$. We define a substitution σ as

$$p\sigma = \bigwedge_{0 \le k \le n, u_k(p) = \mathsf{f}} \mathrm{ed}_{n,k}(\vec{p}).$$

Also, we suppose that $\mathbf{v}((\mathrm{em}_n^0(\bot))) = \mathsf{f}$. Then we have

$$\mathbf{v}(\mathrm{ed}_{n,0}(\vec{p})) = \cdots = \mathbf{v}(\mathrm{ed}_{n,n}(\vec{p})) = \mathsf{f} \tag{1}$$

and

$$\mathbf{v}(\mathrm{ed}_{n,n+1}(\vec{p})) = \cdots = \mathbf{v}(\mathrm{ed}_{n,2^n-1}(\vec{p})) = \mathsf{t}. \tag{2}$$

By (1), for each $k \in \{0, \cdots, n\}$, there exists $v_k \in \mathbf{v}$ such that $v_k(\mathrm{ed}_{n,k}(\vec{p})) = \mathsf{f}$. Also, by the definition of $\mathrm{ed}_{n,k}(\vec{p})$, for each $k \in \{n+1, \cdots, 2^n-1\}$, there exists v_k such that $v_k(\mathrm{ed}_{n,k}(\vec{p})) = \mathsf{f}$. Consequently, we have

$$v_k(\mathrm{ed}_{n,k}(\vec{p})) = \mathsf{f} \text{ for every } k \in \{0, \cdots, 2^n-1\} \tag{3}$$

and

$$\{v_0, \cdots, v_n\} \subseteq \mathbf{v}. \tag{4}$$

By (3) and the definition of $\mathrm{ed}_{n,k}(\vec{p})$, we have

$$v_i(\mathrm{ed}_{n,j}(\vec{p})) = \mathsf{f} \iff i = j,$$

and by the definition of σ, we have

$$u_k(p) = \mathsf{f} \iff v_k(p\sigma) = \mathsf{f},$$

and using induction on χ, we observe

$$u_k(\chi) = v_k(\chi\sigma). \tag{5}$$

By (4) and (5), we have

$$\mathbf{u}(\chi) = \mathsf{f} \text{ implies } \mathbf{v}(\chi\sigma) = \mathsf{f}. \tag{6}$$

So, if we show

$$\mathbf{u}(\chi) = \mathsf{t} \text{ implies } \mathbf{v}(\chi\sigma) = \mathsf{t}, \tag{7}$$

then using (6), we have

$$\mathbf{u}(\chi) = \mathbf{v}(\chi\sigma),$$

and using $\mathbf{u}(I) = \mathsf{f}$, we obtain $\mathbf{v}(I\sigma) = \mathsf{f}$. So, the remaining to be done is to show (7).

Suppose that $\mathbf{u}(\chi) = \mathsf{t}$. Then by (5), we have

$$\{v_0, \cdots, v_n\}(\chi\sigma) = \mathsf{t}. \tag{8}$$

Let v be a member of $\mathbf{v}$. By the definition of v_k, there exists $k \in \{0, \cdots, 2^n - 1\}$ such that

$$v(p_i) = v_k(p_i) \text{ for every } i \in \{1, \cdots, n\}.$$

Therefore, by an induction on χ^*, we observe

$$v(\chi^*) = v_k(\chi^*) \text{ for each } \chi^* \in \mathbf{Wff}(\vec{p}), \tag{9}$$

where $\mathbf{Wff}(\vec{p})$ is the set of formulas in which propositional variables occuring in the formula are only $p_1, \cdots, p_n$. If $k > n$, then by (9) and (3), we have

$$v(\mathrm{ed}_{n,k}(\vec{p})) = v_k(\mathrm{ed}_{n,k}(\vec{p})) = \mathsf{f},$$

which is in contradiction with $v \in \mathbf{v}$ and (2). So, we have $k \leq n$, and by (9) and (8), we obtain

$$v(\chi\sigma) = v_k(\chi\sigma) = \mathtt{t}.$$

Hence, we obtain (7). □

LEMMA 3.5. *If*

$$T, \{\Gamma_i \Rightarrow \bot \mid 1 \in \{1, \cdots, n\}\} \vdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi, \tag{1}$$

then either

$$T, \{\Gamma_i \Rightarrow \psi \mid 1 \in \{1, \cdots, n\}\} \vdash_{\mathbf{Sc}} \Gamma \Rightarrow \psi \tag{2}$$

or

$$T \vdash_{\mathbf{Sc}} \Gamma \Rightarrow \phi. \tag{3}$$

PROOF: We use an induction on a proof figure.

Basis. If $(\Gamma \Rightarrow \phi) \in T$ or $\vdash_{\mathbf{Gc}} \Gamma \Rightarrow \phi$, then we have (3). If $(\Gamma \Rightarrow \phi) = (\Gamma_i \Rightarrow \bot)$, then we have (2).

Induction step. We only consider the following (cut):

$$\frac{\Gamma \Rightarrow \chi \qquad \chi, \Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi}.$$

By applying the induction hypothesis to the left upper sequent, we have either (2) or

$$T \vdash_{\mathbf{Sc}} \Gamma \Rightarrow \chi. \tag{4}$$

So, we assume that (4) holds. Also, to the right upper sequent, we have either

$$T, \{\Gamma_i \Rightarrow \psi \mid 1 \in \{1, \cdots, n\}\} \vdash_{\mathbf{Sc}} (\chi, \Gamma \Rightarrow \psi) \tag{5}$$

or

$$T \vdash_{\mathbf{Sc}} (\chi, \Gamma \Rightarrow \phi). \tag{6}$$

If (5) holds, then by (4), we have (2); if (6) holds, then by (4), we have (3). □

LEMMA 3.6. *Let I_1 and I_2 be inference rules defined below:*

$$\frac{X_1 \quad \cdots \quad X_m}{X} I_1 \qquad \frac{X_1 \quad \cdots \quad X_m \quad (\mathbf{suc}(X), \mathbf{ant}(X) \Rightarrow \psi)}{\mathbf{ant}(X) \Rightarrow \psi} I_2,$$

where ψ does not occur in I_1. Then

$$T \vdash_{\mathbf{Sc}+I_1} Y \iff T \vdash_{\mathbf{Sc}+I_2} Y.$$

PROOF: By the figure

$$\frac{X_1 \quad \cdots \quad X_m \quad (\mathbf{suc}(X), \mathbf{ant}(X) \Rightarrow \mathbf{suc}(X))}{X} I_2,$$

we can see that I_1 holds in $\vdash_{\mathbf{Sc}+I_2}$. Also, by the figure

$$\frac{\dfrac{X_1 \quad \cdots \quad X_m}{X} I_1 \qquad \mathbf{suc}(X), \mathbf{ant}(X) \Rightarrow \psi}{\mathbf{ant}(X) \Rightarrow \psi} (\mathrm{cut})$$

we can see that I_2 holds in $\vdash_{\mathbf{Sc}+I_1}$. □

For example, we apply the above lemma for the following I_1 and I_2:

$$\frac{\phi_1, \Gamma \Rightarrow \phi_2}{\Gamma \Rightarrow \phi_1 \to \phi_2} I_1 \qquad \frac{\phi_1, \Gamma \Rightarrow \phi_2 \quad \phi_1 \to \phi_2, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \psi} I_2.$$

Also, for the following I_1 and I_2

$$\frac{\phi_1, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow \neg\phi_1} I_1 \qquad \frac{\phi_1, \Gamma \Rightarrow \bot \quad \neg\phi_1, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \psi} I_2.$$

So, by Lemma 3.6, we have only to consider the schema of inference rules whose succedent of the lower sequent does not contain logical connectives.

LEMMA 3.7. *Let I be an adequate inference rule. If there exists $\mathbf{u}$ such that $\#(\mathbf{u}) = n + 1$ and $\mathbf{u}(I) = \mathsf{f}$, then (em_n^*) holds in $\vdash_{\mathbf{Sc}+I}$.*

PROOF: It is sufficient to show that (em_n^0) holds in $\vdash_{\mathbf{Sc}+I}$.

We define $T_u(\psi)$ and T_u as the following sets of upper sequents of (em_n^0):

$$T_u(\psi) = \{(\mathrm{ed}_{n,k}(\vec{p}), \Gamma \Rightarrow \psi) \mid k \in \{0, \cdots, n\}\},$$

$$T_u = \{\Gamma \Rightarrow \mathrm{ed}_{n,k}(\vec{p}) \mid k \in \{n+1, \cdots, 2^n - 1\}\}.$$

By Lemma 3.6, we can assume that I is of the form

$$\frac{X_1 \quad \cdots \quad X_{m+1}}{X} I,$$

where $\mathbf{suc}(X) = \mathbf{suc}(X_{m+1}) = \chi$ and χ only occurs in the succedents of X and X_{m+1}. We consider the following instance $I(\bot)$ of I:

$$\frac{X_1 \quad \cdots \quad X_m \quad \mathbf{ant}(X_{m+1}) \Rightarrow \bot}{\mathbf{ant}(X) \Rightarrow \bot} I(\bot),$$

We note that for each $\mathbf{v}$,

$$\mathbf{v}(I) = \mathsf{f} \text{ implies } \mathbf{v}(I(\bot)) = \mathsf{f},$$

and using $\mathbf{u}(I) = \mathsf{f}$, we have $\mathbf{u}(I(\bot)) = \mathsf{f}$. Using Lemma 3.4, there exists a substitution σ such that, for each $\mathbf{v}$,

$$\mathbf{v}(I(\bot)\sigma) = \mathsf{t} \text{ implies } \mathbf{v}((\mathrm{em}_n^0(\bot))) = \mathsf{t}.$$

Therefore, we have

- $\mathbf{v}(T_u(\bot)) = \mathbf{v}(T_u) = \mathbf{v}(\Gamma) = \mathbf{v}(\mathbf{ant}(X)\sigma \Rightarrow \bot) = \mathsf{t}$ implies $\mathbf{v}(\bot) = \mathsf{t}$,
- for each $i \in \{1, \cdots, m\}$, $\mathbf{v}(T_u(\bot)) = \mathbf{v}(T_u) = \mathbf{v}(\Gamma) = \mathbf{v}(\mathbf{ant}(X_i)\sigma) = \mathsf{t}$ implies $\mathbf{v}(\mathbf{suc}(X_i)\sigma) = \mathsf{t}$,
- $\mathbf{v}(T_u(\bot)) = \mathbf{v}(T_u) = \mathbf{v}(\Gamma) = \mathbf{v}(\mathbf{ant}(X_{m+1})\sigma) = \mathsf{t}$ implies $\mathbf{v}(\bot) = \mathsf{t}$.

By Lemma 1.8, we have

- $T_u(\bot) \cup T_u \cup \{\Gamma, \mathbf{ant}(X)\sigma \Rightarrow \bot\} \vdash_{\mathbf{Sc}} \Gamma \Rightarrow \bot$,
- for each $i \in \{1, \cdots, m\}$, $T_u(\bot) \cup T_u \vdash_{\mathbf{Sc}} (\Gamma, \mathbf{ant}(X_i)\sigma \Rightarrow \mathbf{suc}(X_i)\sigma)$,
- $T_u(\bot) \cup T_u \vdash_{\mathbf{Sc}} (\Gamma, \mathbf{ant}(X_{m+1})\sigma \Rightarrow \bot)$.

By Lemma 3.5, we have

(1) $T_u(\psi) \cup T_u \cup \{\Gamma, \mathbf{ant}(X)\sigma \Rightarrow \psi\} \vdash_{\mathbf{Sc}} \Gamma \Rightarrow \psi$,

(2) for each $i \in \{1, \cdots, m\}$, either $T_u(\psi) \cup T_u \vdash_{\mathbf{Sc}} (\Gamma, \mathbf{ant}(X_i)\sigma \Rightarrow \psi)$ or $T_u \vdash_{\mathbf{Sc}} (\Gamma, \mathbf{ant}(X_i)\sigma \Rightarrow \mathbf{suc}(X_i)\sigma)$,

(3) $T_u(\psi) \cup T_u \vdash_{\mathbf{Sc}} (\Gamma, \mathbf{ant}(X_{m+1})\sigma \Rightarrow \psi)$.

Since I is adequate, for each $\chi'_i \in \{\mathbf{suc}(X_i)\sigma, \psi\}$, the following rule also holds in $\vdash_{\mathbf{Sc}+I}$:

$$\frac{(\Gamma, \mathbf{ant}(X_1)\sigma \Rightarrow \chi_1') \ \cdots \ (\Gamma, \mathbf{ant}(X_m)\sigma \Rightarrow \chi_m') \quad (\Gamma, \mathbf{ant}(X_{m+1})\sigma \Rightarrow \psi)}{\Gamma, \mathbf{ant}(X)\sigma \Rightarrow \psi}.$$

Applying the above rule to (2) and (3), we have

$$T_u(\psi) \cup T_u \vdash_{\mathbf{Sc}+I} (\Gamma, \mathbf{ant}(X)\sigma \Rightarrow \psi),$$

and using (1), we have

$$T_u(\psi) \cup T_u \vdash_{\mathbf{Sc}+I} \Gamma \Rightarrow \psi.$$

Hence, (em_n^0) holds in $\vdash_{\mathbf{Sc}+I}$. □

LEMMA 3.8. *Let I be an adequate inference rule and let n be the minimum number of $\{\#(\mathbf{v}) \mid \mathbf{v}(I) = \mathsf{f}\}$. If $n > 1$, then*

$$T \vdash_{\mathbf{Sc}+I} X \iff T \vdash_{n-1} X.$$

PROOF: Since n is minimum, we have

$$\#(\mathbf{v}) \le n - 1 \text{ implies } \mathbf{v}(I) = \mathsf{t}, \text{ for every } \mathbf{v} \tag{1}$$

and

$$\#(\mathbf{v}) = n \text{ and } \mathbf{v}(I) = \mathsf{f} \text{ for some } \mathbf{v} \tag{2}$$

By (1) and Lemma 2.8, we obtain that I holds in $\vdash_{n-1}$. By (2) and Lemma 3.3 and Lemma 3.7, we obtain that (em_{n-1}) holds in $\vdash_{\mathbf{Sc}+I}$. □

By Lemma 3.8, we obtain Theorem 3.1(2).

Acknowledgements. The author would like to thank anonymous referees for their valuable comments.

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