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A 2 SET-UP BINARY ROUTLEY SEMANTICS FOR GÖDELIAN 3-VALUED LOGIC G3 AND ITS PARACONSISTENT COUNTERPART $G3_{\bar{Ł}}^{\leq}$

Abstract

G3 is Gödelian 3-valued logic, $G3_{\bar{Ł}}^{\leq}$ is its paraconsistent counterpart and $G3_{Ł}^{1}$ is a strong extension of $G3_{\bar{Ł}}^{\leq}$. The aim of this paper is to endow each one of the logics just mentioned with a 2 set-up binary Routley semantics.

Keywords: Binary Routley semantics, 2 set-up binary Routley semantics, 3-valued logics, paraconsistent logics, Gödelian 3-valued logic G3.

1. Introduction

The aim of this paper is to define a 2 set-up binary Routley semantics (2bR-semantics) for each one of the logics G3, $G3_{\bar{Ł}}^{\leq}$ and $G3_{Ł}^{1}$. G3 is Gödelian 3-valued logic (cf. [3]), $G3_{\bar{Ł}}^{\leq}$ is the paraconsistent counterpart to G3 and $G3_{Ł}^{1}$ is a strong extension of $G3_{\bar{Ł}}^{\leq}$. The logics $G3_{\bar{Ł}}^{\leq}$ and $G3_{Ł}^{1}$ were introduced in [6]. Proof-theoretically, they were defined as Hilbert-type systems. Semantically, "two-valued" Belnap-Dunn semantics was the tool to interpret them. Nevertheless, they were endowed with a general Routley-Meyer semantics in [4] and with a binary Routley one in [7]. Recently, Avron (cf. [1]) has provided Gentzen-type calculi equivalent to the Hilbert-type formulations for $G3_{\bar{Ł}}^{\leq}$ and $G3_{Ł}^{1}$ defined in [6].

2 set-up Routley-Meyer semantics (2RM-semantics) is introduced in [2], where the logics BN4, RM3 and Łukasiewicz's 3-valued logic Ł3 are interpreted with said semantics. Additionally, the logic E4 is also given a

2RM-semantics in [5]. 2RM-semantics is a particular class of the general Routley-Meyer semantics (cf. [10, Chapter 4]) adequate for interpreting some finite many-valued logics. 2RM-models are based upon structures of the type $(K, R, *)$, where K is a 2 set-up set, $*$ is the Routley operator and R is the ternary relation on K characteristic of the general Routley-Meyer semantics.

On the other hand, 2 set-up binary Routley semantics (2bR-semantics) is going to be introduced for the first time in the present paper, to the best of our knowledge. As it is the case with general Routley-Meyer semantics and 2RM-semantics, 2bR-semantics is a particular class of general binary Routley semantics, introduced in [7]. 2bR-semantics is adequate for interpreting some finite many-valued logics. 2bR-models are based upon structures of the type $(K, R, *)$, where K and $*$ are defined similarly as in 2RM-semantics, but R is a binary relation on K, instead of a ternary one.

It is our opinion that a semantic interpretation S of a given logic L alternative to the standard one, especially if it is a simple one, as it is the case with 2bR-semantics, sheds new light not only on the alternatively interpreted logic L, but also on the connection between L and the class of logics SL interpreted with S, as well as on the elements of the class SL itself. In this regard, we hope that the 2bR-semantics for G3, $G3_{\text{Ł}}^{\leq}$ and $G3_{\text{Ł}}^1$ introduced in the present paper will be useful in the sense just explained, but also in illustrating how the much discussed Routley-Meyer semantics (cf., e.g., [8] and the references therein) behave in the simple setting of a two-element model.

The structure of the paper is as follows. In Section 2, the definition of the logics $G3_{\text{Ł}}^{\leq}$, $G3_{\text{Ł}}^1$ and G3 is recalled. In Section 3, $G3_{\text{Ł}}^{\leq}$ is given a 2bR-semantics (a $2bRG3_{\text{Ł}}^{\leq}$-semantics) and the (strong) soundness theorem w.r.t. $2bRG3_{\text{Ł}}^{\leq}$-semantics is proved. In Section 4, it is shown that $G3_{\text{Ł}}^{\leq}$ is (strongly) complete w.r.t. $2bRG3_{\text{Ł}}^{\leq}$-semantics by using a proof based upon a canonical model construction. In Section 5, (resp., Section 6), we give a $2bRG3_{\text{Ł}}^1$-semantics (resp., a 2bRG3-semantics) for $G3_{\text{Ł}}^1$ (resp., G3). Then, the results in Section 3 and Section 4 are essentially used to prove (strong) soundness and completeness theorems for $G3_{\text{Ł}}^1$ and G3 w.r.t. their respective 2bR-semantics.

2. The logics G3$_{\text{Ł}}^{\leq}$, G3$_{\text{Ł}}^{1}$ and G3

In this section, the logics G3$_{\text{Ł}}^{\leq}$, G3$_{\text{Ł}}^{1}$ and G3 are defined. Firstly, some preliminary notions are noted. Then, we define the matrices MG3$_{\text{Ł}}$ and MG3.

DEFINITION 2.1 (Some preliminary notions). The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ...$, and some or all of the following connectives: $\rightarrow$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of formulas (wffs) are defined in the customary way. A, B, etc, are metalinguisitic variables. Logics are formulated as Hilbert-type axiomatic systems, the notions of "theorem" and "proof from a set of premises" being the usual ones, while the following notions are understood in a fairly standard sense (cf., e.g., [9]): extension and expansion of a given logic; logical matrix M and M-interpretation, M-consequence and M-validity and finally, M-determined logic.

DEFINITION 2.2 (The matrices MG3$_{\text{Ł}}$ and MG3). The matrix MG3$_{\text{Ł}}$ is the structure $(\mathcal{V}, D, \mathbf{F})$ where (1) $\mathcal{V}$ is $\{0, \frac{1}{2}, 1\}$ with $0 < \frac{1}{2} < 1$; (2) $D = \{1\}$; (3) $\mathbf{F} = \{f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg}\}$ where $f_{\wedge}$ and $f_{\vee}$ are defined as the glb (or lattice meet) and the lub (or lattice joint), respectively, and $f_{\neg}$ is an involution with $f_{\neg}(1) = 0, f_{\neg}(0) = 1, f_{\neg}(\frac{1}{2}) = (\frac{1}{2})$, while $f_{\rightarrow}$ is defined according to the following truth-table (tables for $\wedge, \vee$ and $\neg$ are also displayed):

$\rightarrow$	0	$\frac{1}{2}$	1		$\wedge$	0	$\frac{1}{2}$	1		$\vee$	0	$\frac{1}{2}$	1			$\neg$
0	1	1	1		0	0	0	0		0	0	$\frac{1}{2}$	1		0	1
$\frac{1}{2}$	0	1	1		$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1		$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1		1	0	$\frac{1}{2}$	1		1	1	1	1		1	0

Then, MG3 is defined exactly as MG3$_{\text{Ł}}$, except that $f_{\neg}$ is now interpreted according to the following truth-table:

	$\neg$
0	1
$\frac{1}{2}$	0
1	0

Well then, the logic $G3_{\text{Ł}}^{\leq}$ (resp., $G3_{\text{Ł}}^{1}$) is determined by the degree of truth-preserving (resp., truth-preserving) consequence relation defined on the matrix $MG3_{\text{Ł}}$. On the other hand, Gödelian 3-valued logic G3 is determined by the truth-preserving consequence relation defined on the matrix MG3 (cf. [6] and references therein).

The logics $G3_{\text{Ł}}^{\leq}$ and $G3_{\text{Ł}}^{1}$ are expansions of positive intuitionistic logic H_{+}, while G3 is an extension of intuitionistic logic H. They are defined as follows (cf. [6], [7] and references therein).

DEFINITION 2.3 (The logic $G3_{\text{Ł}}^{\leq}$). The logic $G3_{\text{Ł}}^{\leq}$ can be axiomatized as follows:

A1. $A \rightarrow (B \rightarrow A)$

A2. $[A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$

A3. $(A \wedge B) \rightarrow A$; $(A \wedge B) \rightarrow B$

A4. $A \rightarrow [B \rightarrow (A \wedge B)]$

A5. $A \rightarrow (A \vee B)$; $B \rightarrow (A \vee B)$

A6. $(A \rightarrow C) \rightarrow [(B \rightarrow C) \rightarrow [(A \vee B) \rightarrow C)]]$

A7. $A \rightarrow \neg\neg A$

A8. $\neg\neg A \rightarrow A$

A9. $(A \vee \neg B) \vee (A \rightarrow B)$

A10. $\neg A \rightarrow [A \vee (A \rightarrow B)]$

A11. $(A \wedge \neg A) \rightarrow (B \vee \neg B)$

Rules

Modus Ponens (MP): If $A \rightarrow B$ and A, then B.
Contraposition (Con): If $A \rightarrow B$ is a theorem, then $\neg B \rightarrow \neg A$ is also
a theorem.

Remark 2.4 (Rules of inference and rules of proof). A rule r of a logic L is a 'rule of inference' if it can be applied to any premises formulated in the language of L; and r is a 'rule of proof' if it is applied only to theorems of L. Notice that Con is formulated as a rule of proof in $G3_{\text{Ł}}^{\leq}$ (cf. [6,

Remark 6.23], [8, §1.5] on this important question in logics with weak rules of inference).

DEFINITION 2.5 (The logic G3$_\mathrm{L}^1$). The logic G3$_\mathrm{L}^1$ is defined exactly as G3$_\mathrm{L}^\leq$ except that now Con is understood as a rule of inference: If $A \to B$, then $\neg B \to \neg A$.

DEFINITION 2.6 (The logic G3). The logic G3 is axiomatized by adding

$$\text{A12. } (A \to B) \to (\neg B \to \neg A)$$
$$\text{A13. } \neg A \to (A \to B)$$

to A1–A7 and A9 of G3$_\mathrm{L}^\leq$. The sole rule of inference is MP (cf. [7, §A2]).

The section is ended by noting some theorems and rules of the logics just defined.

Remark 2.7 (Some theorems and rules of G3$_\mathrm{L}^\leq$, G3$_\mathrm{L}^1$ and G3). The following are provable in the three logics defined above (cf. [6, 7] and references therein):

T1. $A \to A$

T2. $[(A \to B) \wedge A] \to B$

T3. $(\neg A \wedge \neg B) \to \neg(A \vee B)$

T4. $\neg B \to [\neg A \vee \neg(A \to B)]$

T5. $\neg(A \to B) \to \neg B$

T6. $[\neg(A \to B) \wedge \neg A] \to A$

T7. $[\neg(A \to B) \wedge (\neg A \wedge B)] \to C$

Efq. If $\neg A$ is a theorem, then $A \to B$ is also a theorem.

In addition, the rule Ecq ("E contradictione quodlibet" —"Any proposition is derivable from a contradiction"), if $A \wedge \neg A$, then B, is provable in G3$_\mathrm{L}^1$, whereas A10 and A11 of G3$_\mathrm{L}^\leq$ and Ecq are, of course, provable in G3. (Efq abbreviates "E falso quodlibet": "Any proposition follows from a false proposition").

3. A 2 set-up binary Routley semantics for G3$_{\overline{\text{Ł}}}^{\leq}$

In this section, G3$_{\overline{\text{Ł}}}^{\leq}$ is given a 2 set-up binary Routley semantics (2bRG3$_{\overline{\text{Ł}}}^{\leq}$-semantics, for short). Firstly, we define the concept of a model and related notions.

DEFINITION 3.1 (2bRG3$_{\overline{\text{Ł}}}^{\leq}$-models). Let $*$ be an involutive unary operation defined on the set K. That is, for any $x \in K$, $x = x^{**}$, and let K be the two-element set $\{0, 0^*\}$. A 2 set-up binary Routley G3$_{\overline{\text{Ł}}}^{\leq}$-model (2bRG3$_{\overline{\text{Ł}}}^{\leq}$-model, for short) is a structure $(K, R, *, \vDash)$ where (I) R is a reflexive binary relation on K such that $R00^*$ or $R0^*0$, and (II) $\vDash$ is a valuation relation from K to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable p, wffs A, B and $a \in K$:

(i) $(Rab \ \& \ a \vDash p) \Rightarrow b \vDash p$

(ii) $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$

(iii) $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$

(iv) $a \vDash A \rightarrow B$ iff for all $b \in K$, $(Rab$ and $b \vDash A) \Rightarrow b \vDash B$

(v) $a \vDash \neg A$ iff $a^* \nvDash A$

DEFINITION 3.2 (2bRG3$_{\overline{\text{Ł}}}^{\leq}$-consequence, 2bRG3$_{\overline{\text{Ł}}}^{\leq}$-validity). For any non-empty set of wffs Γ and wff A, $\Gamma \vDash_M A$ (A is a consequence of Γ in the 2bRG3$_{\overline{\text{Ł}}}^{\leq}$-model M) iff for all $a \in K$ in M, $a \vDash_M A$ whenever $a \vDash_M \Gamma$ ($a \vDash_M \Gamma$ iff $a \vDash_M B$ for all $B \in \Gamma$). Then, $\Gamma \vDash_{\text{2bRG3}_{\overline{\text{Ł}}}^{\leq}} A$ (A is a 2bRG3$_{\overline{\text{Ł}}}^{\leq}$-consequence of Γ) iff $\Gamma \vDash_M A$ for each 2bRG3$_{\overline{\text{Ł}}}^{\leq}$-model M. In particular, if $\Gamma = \emptyset$, $\vDash_M A$ (A is true in M) iff $a \vDash_M A$ for all $a \in K$ in M. And $\vDash_{\text{2bRG3}_{\overline{\text{Ł}}}^{\leq}} A$ (A is 2bRG3$_{\overline{\text{Ł}}}^{\leq}$-valid) iff $\vDash_M A$ in every 2bRG3$_{\overline{\text{Ł}}}^{\leq}$-model.

We prove some facts about 2bRG3$_{\overline{\text{Ł}}}^{\leq}$-models.

PROPOSITION 3.3 ($0^* \vDash \neg A$ iff $0 \nvDash A$). For any 2bRG3$_{\overline{\text{Ł}}}^{\leq}$-model M and wff A, $0^* \vDash_M \neg A$ iff $0 \nvDash_M A$.

PROOF: Immediate by clause (v) in Definition 3.1 and the involutiveness of $*$: $0^* \vDash_M \neg A$ iff (clause (v)) $0^{**} \nvDash_M A$ iff (involutiveness of $*$) $0 \nvDash_M A$. □

LEMMA 3.4 (Hereditary Condition). *For any $2bRG3_{\bar{L}}^{\leq}$-model M, $a, b \in K$ in M and wff A, (Rab & $a \vDash_M A$) $\Rightarrow b \vDash_M A$.*

PROOF: Induction on the structure of A. If A is $B \wedge C$ or $B \vee C$, the proof is immediate. Then, let us prove the cases where A is $B \to C$ and $\neg B$. If $a = b$, the proof is trivial. So, we assume $a \neq b$ (clauses (iv) and (v) in Definition 3.1 are applied without mentioning them). (I) A is $B \to C$. (Ia) $a = 0$ and $b = 0^*$. Suppose then (1) $R00^*$ and (2) $0 \vDash_M B \to C$. We have to prove $0^* \vDash_M B \to C$. There are two possibilities to consider: $R0^*0^*$ and $R0^*0$. Suppose the first one, that is (3) $R0^*0^*$. Assume also (4) $0^* \vDash_M B$. By 1, 2 and 4, we get (5) $0^* \vDash_M C$, as required. Suppose now the second alternative, that is, (6) $R0^*0$. Assume also (7) $0 \vDash_M B$. By reflexivity of R, we have (8) $R00$, whence by 2 and 7, we get (9) $0 \vDash_M C$, as it was to be proved. (Ib) $a = 0^*$ and $b = 0$. Suppose (1) $R0^*0$ and (2) $0^* \vDash_M B \to C$. We have to prove $0 \vDash_M B \to C$. There are two possibilities to consider: $R00$ and $R00^*$. Then, the proof proceeds similarly as in case Ia. (II) A is $\neg B$. (IIa) $a = 0$ and $b = 0^*$. Suppose then (1) $R00^*$ and (2) $0 \vDash_M \neg B$ (i.e., $0^* \nvDash_M B$). By the induction hypothesis, 1 and 2, we have (3) $0 \nvDash B$, i.e., $0^* \vDash \neg B$, by Proposition 3.3, as required. (IIb) $a = 0^*$ and $b = 0$. The proof is similar to that of IIa. $\square$

LEMMA 3.5 (Entailment Lemma). *For any wffs A, B, $\vDash_{2bRG3_{\bar{L}}^{\leq}} A \to B$ iff $(a \vDash_M A \Rightarrow a \vDash_M B$, for all $a \in K$ in all $2bRG3_{\bar{L}}^{\leq}$-models M).*

PROOF: ($\Rightarrow$) Let M be a $2bRG3_{\bar{L}}^{\leq}$-model. Suppose (1) $\vDash_{2bRG3_{\bar{L}}^{\leq}} A \to B$ and (2) $0 \vDash_M A$ (resp., $0^* \vDash_M A$). By reflexivity of R, we have (3) $R00$ and $R0^*0^*$. By 1, 2 and 3, we get (4) $0 \vDash_M B$ (resp., $0^* \vDash_M B$) as desired. ($\Leftarrow$) Suppose (1) $a \vDash_M A \Rightarrow a \vDash_M B$, for all $a \in K$ in M. Furthermore, suppose (2) $R0b$ (resp., $R0^*b$) and $b \vDash_M A$ for a given $b \in K$. Then (3) $b \vDash_M B$ trivially follows from 1, as it was required. $\square$

Now, we can prove soundness of $G3_{\bar{L}}^{\leq}$ w.r.t. $2bRG3_{\bar{L}}^{\leq}$-semantics.

THEOREM 3.6 (Soundness of $G3_{\bar{L}}^{\leq}$). *For any set of wffs Γ and wff A, if $\Gamma \vdash_{G3_{\bar{L}}^{\leq}} A$, then $\Gamma \vDash_{2bRG3_{\bar{L}}^{\leq}} A$.*

PROOF: If $A \in \Gamma$, the proof is trivial; and if A has been obtained by applying MP, the proof is immediate by leaning upon the reflexivity of R.

Then, suppose that A has been obtained by an application of Con. In this case, A is of the form (1) $\neg B \rightarrow \neg C$ and, by hypothesis, we have (2) $\vDash_{2\mathrm{bRG3}_{\mathrm{L}}^{\leq}} C \rightarrow B$. We need to prove $\vDash_{2\mathrm{bRG3}_{\mathrm{L}}^{\leq}} \neg B \rightarrow \neg C$. We use the Entailment Lemma. So, suppose for any arbitrary $2\mathrm{bRG3}_{\mathrm{L}}^{\leq}$-model M, (3) $0 \vDash_{\mathrm{M}} \neg B$ (resp., $0^* \vDash_{\mathrm{M}} \neg B$). By clause (v) (resp., Proposition 3.3), we have (4) $0^* \nvDash_{\mathrm{M}} B$ (resp., $0 \nvDash_{\mathrm{M}} B$), whence by the Entailment Lemma and 2, we get (5) $0^* \nvDash_{\mathrm{M}} C$ (resp., $0 \nvDash_{\mathrm{M}} C$) and (6) $0 \vDash_{\mathrm{M}} \neg C$ (resp., $0^* \vDash_{\mathrm{M}} \neg C$) by applying again clause (v) (resp. Proposition 3.3).

Concerning the axioms, we focus on the characteristic $\mathrm{MG3}_{\mathrm{L}}$-axioms, that is, A9, A10 and A11. The proof of the validity of A1-A6 as well as that of the double negation axioms A7 and A8 is left to the reader (notice that A7 and A8 are immediate by involutiveness of $*$).

A9, $(A \vee \neg B) \vee (A \rightarrow B)$, is $2\mathrm{bRG3}_{\mathrm{L}}^{\leq}$-valid. Suppose that M is a $2\mathrm{bRG3}_{\mathrm{L}}^{\leq}$-model falsifying A9. Then, for some wffs A, B, either (I) $0 \nvDash_{\mathrm{M}} (A \vee \neg B) \vee (A \rightarrow B)$ or (II) $0^* \nvDash_{\mathrm{M}} (A \vee \neg B) \vee (A \rightarrow B)$. Case I: We have (1) $0 \nvDash A$, (2) $0 \nvDash \neg B$ (i.e., $0^* \vDash B$) and (3) $0 \nvDash A \rightarrow B$. There are two possibilities to consider: (4) $R00$, $0 \vDash A$ and $0 \nvDash B$; and (5) $R00^*$, $0^* \vDash A$ and $0^* \nvDash B$. But 4 contradicts 1, while 5 contradicts 2. Case (II) We have (1) $0^* \nvDash A$, (2) $0^* \nvDash \neg B$ (i.e., $0 \vDash B$) and (3) $0^* \nvDash A \rightarrow B$. There are two possibilities to consider: (4) $R0^*0^*$, $0^* \vDash A$ and $0^* \nvDash B$; and (5) $R0^*0$, $0 \vDash A$ and $0 \nvDash B$. But 4 contradicts 1 whereas 5 contradicts 2.

A10, $\neg A \rightarrow [A \vee (A \rightarrow B)]$, is $2\mathrm{bRG3}_{\mathrm{L}}^{\leq}$-valid. Suppose that M is a $2\mathrm{bRG3}_{\mathrm{L}}^{\leq}$-model falsifying A10. By the Entailment Lemma, for some wffs A, B, either (I) $0 \vDash_{\mathrm{M}} \neg A$ and $0 \nvDash_{\mathrm{M}} A \vee (A \rightarrow B)$ or (II) $0^* \vDash_{\mathrm{M}} \neg A$ and $0^* \nvDash_{\mathrm{M}} A \vee (A \rightarrow B)$. Case I: We have (1) $0^* \nvDash A$, (2) $0 \nvDash A$ and (3) $0 \nvDash A \rightarrow B$. Now, either (4) $R00$, $0 \vDash A$ and $0 \nvDash B$ or (5) $R00^*$, $0^* \vDash A$ and $0^* \nvDash B$. But 4 contradicts 2, and 5 contradicts 1. Case II is treated similarly.

A11, $(A \wedge \neg A) \rightarrow (B \vee \neg B)$, is $2\mathrm{bRG3}_{\mathrm{L}}^{\leq}$-valid. The proof is similar to that of A10. □

4. Completeness of $\mathrm{G3}_{\mathrm{L}}^{\leq}$

Completeness of $\mathrm{G3}_{\mathrm{L}}^{\leq}$ is proved by using a canonical model construction. We begin by defining the notion of a $\mathrm{G3}_{\mathrm{L}}^{\leq}$-theory and the classes of $\mathrm{G3}_{\mathrm{L}}^{\leq}$-theories of interest in the present paper.

DEFINITION 4.1 (G3$_{\text{Ł}}^{\leq}$-theories. Classes of G3$_{\text{Ł}}^{\leq}$-theories). A G3$_{\text{Ł}}^{\leq}$-theory (theory, for short) is a set of formulas containing all G3$_{\text{Ł}}^{\leq}$-theorems and closed under Modus Ponens (MP). Let t be a theory. We set: (1) t is prime iff whenever $A \vee B \in t$, then $A \in t$ or $B \in t$; (2) t is trivial iff it contains all wffs; (3) t is a-consistent ('consistent in an absolute sense') iff t is not trivial; (4) t is w-inconsistent ('inconsistent in a weak sense') iff $\neg A \in t$, A being a G3$_{\text{Ł}}^{\leq}$-theorem; then t is w-consistent ('consistent in a weak sense') iff t is not w-inconsistent; (5) t is inconsistent iff $A \wedge \neg A \in t$ for some wff A; then t is consistent if it is not inconsistent (cf. [8] and references therein on the notion of w-consistency).

LEMMA 4.2 (Extension to prime theories). *Let t be a theory and A a wff such that $A \notin t$. Then, there is a prime theory u such that $t \subseteq u$ and $A \notin u$.*

PROOF: We extend t to a maximal theory u such that $A \notin u$. If u is not prime, then there are wffs B, C such that $B \vee C \in u$ but $B \notin u$ and $C \notin u$. Then, we define the sets $[u, B] = \{D \mid B \rightarrow D \in u\}$, $[u, C] = \{D \mid C \rightarrow D \in u\}$. By using A2, it is shown that (1) $[u, B]$ and $[u, C]$ are closed under MP; by using A1, (2) that they include u. Finally, by T1, (3) that $B \in [u, B]$ and $C \in [u, C]$. Next, by the hypothesis and (1), it follows that neither $[u, B]$ nor $[u, C]$ is included in u, whence we have $A \in [u, B]$ and $A \in [u, C]$ due to the maximality of u. But then, we have (4) $A \in u$ by A6 and the fact that $B \vee C \in u$, contradicting our hypothesis. Consequently, u is prime.   □

In what follows, it is shown how canonical 2bRG3$_{\text{Ł}}^{\leq}$-models are built, Also, we prove some general facts about them.

Let Γ be a set of wffs and A a wff such that $\Gamma \nvdash_{\text{G3}_{\text{Ł}}^{\leq}} A$. Then, A is not included in the set of consequences derivable from Γ (in symbols, $A \notin Cn\Gamma[\text{G3}_{\text{Ł}}^{\leq}]$). By the Extension Lemma, there is a prime theory $\mathcal{T}$ such that $Cn\Gamma[\text{G3}_{\text{Ł}}^{\leq}] \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. (Notice that $\mathcal{T}$ is a-consistent.) Then, the canonical 2bRG3$_{\text{Ł}}^{\leq}$-model built upon $\mathcal{T}$ is defined as follows.

DEFINITION 4.3 (Canonical 2bRG3$_{\text{Ł}}^{\leq}$-models). The canonical 2bRG3$_{\text{Ł}}^{\leq}$-model built upon $\mathcal{T}$, as this theory has been defined above, is the structure $(K^C, R^C, *^C, \vDash^C)$, where (1) $K^C = \{\mathcal{T}, \mathcal{T}^{*^C}\}$ and for any wffs A, B and

$a, b \in K^C$, we have: (2) $R^C ab$ iff $(A \to B \in a$ & $A \in b) \Rightarrow B \in b$; (3) $a^{*^C} = \{A \mid \neg A \notin a\}$ and (4) $a \vDash^C A$ iff $A \in a$.

We prove some significant and useful facts about canonical 2bRG3$_{\bar{L}}^{\leq}$-models. By $\mathcal{T}$, we refer to the G3$_{\bar{L}}^{\leq}$-theory upon which each canonical 2bRG3$_{\bar{L}}^{\leq}$-model is built (the superscript C above R and $*$ is dropped when there is no risk of confusion).

PROPOSITION 4.4 ($\mathcal{T}$ is a w-consistent G3$_{\bar{L}}^{\leq}$-theory). The G3$_{\bar{L}}^{\leq}$-theory $\mathcal{T}$ is a w-consistent G3$_{\bar{L}}^{\leq}$-theory.

PROOF: Suppose $\neg A \in \mathcal{T}$, A being a G3$_{\bar{L}}^{\leq}$-theorem. By the rule Efq, $\neg A \to B$ is a G3$_{\bar{L}}^{\leq}$-theorem where B is an arbitrary wff. Then, $B \in \mathcal{T}$, contradicting the a-consistency of $\mathcal{T}$.   □

PROPOSITION 4.5 ($\mathcal{T}^{*^C}$ is a prime G3$_{\bar{L}}^{\leq}$-theory). The $*^C$-image of $\mathcal{T}$, $\mathcal{T}^{*^C}$, is a prime G3$_{\bar{L}}^{\leq}$-theory.

PROOF: (I) $\mathcal{T}^*$ is closed under MP: Suppose (1) $A \to B \in \mathcal{T}^*$ (i.e., $\neg(A \to B) \notin \mathcal{T}$) and (2) $A \in \mathcal{T}^*$ (i.e., $\neg A \notin \mathcal{T}$) but (3) $B \notin \mathcal{T}^*$ (i.e., $\neg B \in \mathcal{T}$). By using the G3$_{\bar{L}}^{\leq}$-theorem $\neg B \to [\neg A \vee \neg(A \to B)]$ (T4), we have (4) $\neg A \in \mathcal{T}$ or $\neg(A \to B) \in \mathcal{T}$. But 1 and 2 contradict 4. (II) $\mathcal{T}^*$ contains all G3$_{\bar{L}}^{\leq}$-theorems: Let A be a G3$_{\bar{L}}^{\leq}$-theorem such that $A \notin \mathcal{T}^*$. Then, $\neg A \in \mathcal{T}$, contradicting the w-consistency of $\mathcal{T}$. (III) $\mathcal{T}^*$ is prime: Suppose (5) $A \vee B \in \mathcal{T}^*$ (i.e., $\neg(A \vee B) \notin \mathcal{T}$) but (6) $A \notin \mathcal{T}^*$ (i.e., $\neg A \in \mathcal{T}$) and (7) $B \notin \mathcal{T}^*$ (i.e., $\neg B \in \mathcal{T}$). By the G3$_{\bar{L}}^{\leq}$-theorem $(\neg A \wedge \neg B) \to \neg(A \vee B)$ (T3), we have (8) $\neg(A \vee B) \in \mathcal{T}$, contradicting 5.   □

Next, an alternative reading of the canonical accessibility relation is provided together with the proof that R^C is a reflexive relation such that $R^C \mathcal{T} \mathcal{T}^{*^C}$ or $R^C \mathcal{T}^{*^C} \mathcal{T}$. Then, it is shown that $*^C$ is an involutive operation in canonical 2bRG3$_{\bar{L}}^{\leq}$-models. Also, that clauses (i), (ii), (iii) and (v) hold in canonical 2bRG3$_{\bar{L}}^{\leq}$-models.

PROPOSITION 4.6 ($R^C ab$ iff $a \subseteq b$). For any $a, b \in K^C$, $R^C ab$ iff $a \subseteq b$.

PROOF: ($\Rightarrow$) Suppose (1) $R^C ab$ and (2) $A \in a$, and let (3) $B \in b$. By A1 and 2, we have (4) $B \to A \in a$, whence (5) $A \in b$ follows by 1, 3 and 4. ($\Leftarrow$) Suppose (1) $a \subseteq b$. (2) $A \to B \in a$ and (3) $A \in b$. By 1 and 2, we have

(4) $A \to B \in b$. By T2, $[(A \to B) \wedge A] \to B$, 3 and 4, (5) $B \in b$ follows, as it was to be proved. $\square$

PROPOSITION 4.7 ($R^C \mathcal{T} \mathcal{T}^{*C}$ or $R^C \mathcal{T}^{*C} \mathcal{T}$). The canonical relation R^C is a reflexive relation such that $R^C \mathcal{T} \mathcal{T}^{*C}$ or $R^C \mathcal{T}^{*C} \mathcal{T}$.

PROOF: By Proposition 4.6, it is immediate that R^C is reflexive. On the other hand, suppose that there are A, B such that (1) $A \in \mathcal{T}$, (2) $B \in \mathcal{T}^*$ (i.e., $\neg B \notin \mathcal{T}$), (3) $A \notin \mathcal{T}^*$ (i.e., $\neg A \in \mathcal{T}$) and (4) $B \notin \mathcal{T}$. By $(A \wedge \neg A) \to (B \vee \neg B)$ (A11), we have (5) $B \vee \neg B \in \mathcal{T}$. But 2 and 4 contradict 5. $\square$

PROPOSITION 4.8 ($*^C$ is an involutive operation on K^C). The canonical operation $*^C$ is an involutive operation on K^C.

PROOF: Let $a \in K^C$. Given that a is a G3$\frac{\leq}{\text{Ł}}$-theory, $A \in a$ iff $\neg \neg A \in a$ follows by A7 and A8 Then, we have $A \in a$ iff $A \in a^{**}$ by Definition 4.3(3). $\square$

PROPOSITION 4.9 (Clauses (i), (ii), (iii) and (v) hold canonically). Conditions (i), (ii), (iii) and (v) in Definition 3.1 hold when canonically interpreted according to Definition 4.3.

PROOF: Condition (i) is trivial by Proposition 4.6 and condition (v) by Definition 4.3(4). Then, condition (iii) (resp., condition (ii)) is immediate by A5, A6 and primeness of both $\mathcal{T}$ and $\mathcal{T}^*$ (resp., A3 and A4). $\square$

Concerning clause (iv), we have:

PROPOSITION 4.10 (Clause (iv) holds in the canonical 2bRG3$\frac{\leq}{\text{Ł}}$-model). Condition (iv) in Definition 3.1 holds in the canonical 2bRG3$\frac{\leq}{\text{Ł}}$-model.

PROOF: ($\Rightarrow$). Let $a \in K^C$ and suppose $a \vDash^C A \to B$ (i.e., $A \to B \in a$), $R^C ab$ (i.e., $a \subseteq b$) and $b \vDash^C A$ (i.e., $A \in b$). Then, $b \vDash^C B$ (i.e., $B \in b$) is immediate by MP.

($\Leftarrow$) We use Proposition 4.7. (I) $\mathcal{T} \subseteq \mathcal{T}^*$. (Ia) Assume $A \to B \notin \mathcal{T}$ (i.e., $\neg(A \to B) \in \mathcal{T}^*$). Given $R\mathcal{T}\mathcal{T}$ and $R\mathcal{T}\mathcal{T}^*$, it suffices to show $[A \in \mathcal{T} \ \& \ B \notin \mathcal{T}]$ or $[A \in \mathcal{T}^* \ \& \ B \notin \mathcal{T}^*]$. For reductio, suppose (1) $[A \notin \mathcal{T} \ \& \ A \notin \mathcal{T}^*]$ or (2) $[A \notin \mathcal{T} \ \& \ B \in \mathcal{T}^*]$ or (3) $[B \in \mathcal{T} \ \& \ A \notin \mathcal{T}^*]$ or (4) $[B \in \mathcal{T} \ \& \ B \in \mathcal{T}^*]$. But 1, 2, 3 and 4 are impossible by $\neg A \to [A \vee (A \to B)]$ (A10), $(A \vee \neg B) \vee (A \to B)$ (A9), $A \to (B \to A)$ (A1) and A1, respectively. (Ib) Assume $A \to B \notin \mathcal{T}^*$ (i.e., $\neg(A \to B) \in \mathcal{T}$). Given

$R\mathcal{T}^*\mathcal{T}^*$, it suffices to show $A \in \mathcal{T}^*$ and $B \notin \mathcal{T}^*$. Suppose, for reductio, (1) $A \notin \mathcal{T}^*$ (i.e., $\neg A \in \mathcal{T}$) or (2) $B \in \mathcal{T}^*$ (i.e, $\neg B \notin \mathcal{T}$). By I and 1, (3) $A \notin \mathcal{T}$ follows. But 1 and 2 are impossible by $[\neg(A \to B) \wedge \neg A] \to A$ (T6) and $\neg(A \to B) \to \neg B$ (T5), respectively.

(II) $\mathcal{T}^* \subseteq \mathcal{T}$. (IIa) Assume $A \to B \notin \mathcal{T}$. Given $R\mathcal{T}\mathcal{T}$, it suffices to show $A \in \mathcal{T}$ and $B \notin \mathcal{T}$. By II and IIa, we have (1) $A \to B \notin \mathcal{T}^*$ (i.e., $\neg(A \to B) \in \mathcal{T}$). Suppose now, for reductio, (2) $A \notin \mathcal{T}$ or (3) $B \in \mathcal{T}$. If 3 obtains, then $A \to B \in \mathcal{T}$ is immediate by A1, contradicting IIa. Let then 2 be the case. By II, we have (4) $A \notin \mathcal{T}^*$ (i.e., $\neg A \in \mathcal{T}$). Next, $[\neg(A \to B) \wedge \neg A] \to A$ (T6) is used. By T6, 1 and 4, (5) $A \in \mathcal{T}$ follows, contradicting 2. (IIb) $A \to B \notin \mathcal{T}^*$ (i.e., $\neg(A \to B) \in \mathcal{T}$). Given $R\mathcal{T}^*\mathcal{T}^*$ and $R\mathcal{T}^*\mathcal{T}$, it suffices to show $[A \in \mathcal{T}^*$ & $B \notin \mathcal{T}^*]$ or $[A \in \mathcal{T}$ & $B \notin \mathcal{T}]$. Then, the proof is similar to that of Ia by using now $[\neg(A \to B) \wedge \neg A] \to A$ (T6), $[\neg(A \to B) \wedge (\neg A \wedge B)] \to C$ (T7) and $\neg(A \to B) \to \neg B$ (T5).

We remark that the use of A9 (resp., T7) requires the primeness (resp., the a-consistency) of $\mathcal{T}$. □

Remark 4.11 (On the canonical clause (iv)). Suppose that R is required to be only reflexive: it is not demanded of $2\mathrm{bRG3}_{\mathrm{Ł}}^{\leq}$-models that one of $R00^*$ and $R0^*0$ be present. Then, the proof of the canonical validity of clause (iv) would require the theoremhood of disjunctive Peirce's law, $A \vee (A \to B)$.

Once Proposition 4.10 proved, it immediately follows that the canonical $2\mathrm{bRG3}_{\mathrm{Ł}}^{\leq}$-model is indeed a $2\mathrm{bRG3}_{\mathrm{Ł}}^{\leq}$-model.

LEMMA 4.12 (The canonical model is indeed a model). *The canonical $2bRG3_{Ł}^{\leq}$-model is indeed a $2bRG3_{Ł}^{\leq}$-model.*

PROOF: (1) By Proposition 4.7, R^C is a reflexive relation such that $R\mathcal{T}\mathcal{T}^*$ or $R\mathcal{T}^*\mathcal{T}$. (2) By Proposition 4.8, $*^C$ is an involutive operation on K^C. (3) Finally, by Propositions 4.9 and 4.10, $\vDash^C$ fulfils conditions (i)–(v) in Definition 3.1. □

Now, we prove completeness.

THEOREM 4.13 (Completeness of $\mathrm{G3}_{\mathrm{Ł}}^{\leq}$). *For any set of wffs Γ and wff A, if $\Gamma \vDash_{2bRG3_{Ł}^{\leq}} A$, then $\Gamma \vdash_{G3_{Ł}^{\leq}} A$.*

PROOF: Suppose $\Gamma \nvdash_{\mathrm{G3}_{\mathrm{Ł}}^{\leq}} A$. By the Extension Lemma (Lemma 4.2), there is a prime theory $\mathcal{T}$ such that $\Gamma \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. Then, the canonical $2\mathrm{bRG3}_{\mathrm{Ł}}^{\leq}$-model is defined upon $\mathcal{T}$ as shown in Definition 4.3. By

Lemma 4.12, the canonical $2\text{bRG}3^{\leq}_{\text{Ł}}$-model is a $2\text{bRG}3^{\leq}_{\text{Ł}}$-model. Then, $\Gamma \nvDash^C A$, since $\mathcal{T} \vDash^C \Gamma$ but $\mathcal{T} \nvDash^C A$. Consequently, $\Gamma \nvDash_{2\text{bRG}3^{\leq}_{\text{Ł}}} A$ by Definition 3.2. $\qquad\square$

5. A 2 set-up binary Routley semantics for $\text{G}3^1_{\text{Ł}}$

In this section, $\text{G}3^1_{\text{Ł}}$ is given a 2 set-up binary Routley semantics ($2\text{bRG}3^1_{\text{Ł}}$-semantics, for short) and $\text{G}3^1_{\text{Ł}}$ is proved strongly sound and complete w.r.t. said semantics (we lean upon the results in Sections 3 and 4).

DEFINITION 5.1 ($2\text{bRG}3^1_{\text{Ł}}$-models). A 2-set-up binary $\text{G}3^1_{\text{Ł}}$-model ($2\text{bRG}3^1_{\text{Ł}}$-model, for short) is a structure $(K, R, *, \vDash)$ where $K, *$ and $\vDash$ are defined exactly as in $2\text{bRG}3^{\leq}_{\text{Ł}}$-models and R is a reflexive relation such that $R00^*$, instead of being a reflexive relation such that $R00^*$ or $R0^*0$, as in $2\text{bRG}3^{\leq}_{\text{Ł}}$-models.

DEFINITION 5.2 ($2\text{bRG}3^1_{\text{Ł}}$-consequence, $2\text{bRG}3^1_{\text{Ł}}$-validity). The notions of $2\text{bRG}3^1_{\text{Ł}}$-consequence and $2\text{bRG}3^1_{\text{Ł}}$-validity are defined similarly as the corresponding notions for $\text{G}3^{\leq}_{\text{Ł}}$, except that in each model M they are restricted now to the element 0 in K. Thus, for example, $\Gamma \vDash_{\text{M}} A$ iff $0 \vDash_{\text{M}} A$, whenever $0 \vDash \Gamma$ ($0 \vDash \Gamma$ iff $0 \vDash B$ for all $B \in \Gamma$).

Then, we note that Proposition 3.3 and Lemmas 3.4 and 3.5 still hold for $\text{G}3^1_{\text{Ł}}$ and are proved in a similar way as in $\text{G}3^{\leq}_{\text{Ł}}$.

Concerning soundness, the essential point is to prove that Contraposition (Con) holds as a rule of inference.

PROPOSITION 5.3 (Con preserves $2\text{bRG}3^1_{\text{Ł}}$-validity). Con (if $A \to B$, then $\neg B \to \neg A$) preserves $2\text{bRG}3^1_{\text{Ł}}$-validity.

PROOF: Let M be a $2\text{bRG}3^1_{\text{Ł}}$-model and A, B wffs such that (1) $0 \vDash A \to B$ but (2) $0 \nvDash \neg B \to \neg A$. There are two possibilities to consider: (3) $R00$, $0 \vDash \neg B$ (i.e., $0^* \nvDash B$), $0 \nvDash \neg A$ (i.e., $0^* \vDash A$) and (4) $R00^*$, $0^* \vDash \neg B$ (i.e., $0 \nvDash B$) and $0^* \nvDash \neg A$ (i.e., $0 \vDash A$). If 3 obtains, we get (5) $0^* \vDash B$ by 1, since $R00^*$ holds in M. But 3 and 5 contradict each other. If, on the other hand, 4 is the case. we have (6) $0 \vDash B$ by using again 1, since $R00$ holds in M. But, as in the previous case, a contradiction arises (6 contradicts 4). $\qquad\square$

Remark 5.4 (Con cannot be validated w.r.t. K). We note that if $2\text{bRG}3^1_{\text{Ł}}$-consequence is defined w.r.t. K instead of w.r.t. only 0 in K, Con as a rule

of inference does not preserve $2bRG3_L^1$-validity. Consider a $2bRG3_L^1$-model
M where $R0^*0$ does not follow and, for distinct propositional variables p, q,
we have $0 \vDash_M p$ (i.e., $0^* \nVDash_M \neg p$), $0 \nVDash_M q$ (i.e., $0^* \vDash_M \neg q$), $0^* \vDash_M p$ and
$0^* \vDash_M q$. Clearly, $0^* \vDash_M p \to q$ but $0^* \nVDash_M \neg q \to \neg p$ as $R0^*0^*$ holds by
reflexivity of R. Also, notice that by the Entailment Lemma, this $2bRG3_L^1$-
model shows that the contraposition axiom, $(A \to B) \to (\neg B \to \neg A)$, is
not $2bRG3_L^1$-valid.

Now, the proof that MP preserves $2bRG3_L^1$-validity and that A1-A11
are $2bRG3_L^1$-valid is similar as in $2bRG3_{\bar{L}}^{\leq}$-models. In fact, it is simpler. If
A is an implicative axiom, only the case $R00^*$, not both $R00^*$ and $R0^*0$, as
in $2bRG3_{\bar{L}}^{\leq}$-models, has to be considered. And if A is A9, only truth w.r.t.
0, not w.r.t. both 0 and 0^*, has to be examined. Finally, that MP preserves
$2bRG3_L^1$-validity is immediate by reflexivity of R, as in $2bRG3_{\bar{L}}^{\leq}$-models.

As regards completeness, the main difference w.r.t. $G3_{\bar{L}}^{\leq}$ is that $G3_L^1$-
theories need now to be closed under Con. Consequently, the Extension
Lemma (Lemma 4.2) does not hold, as it stands, in the case of $G3_L^1$. Nev-
ertheless, the disjunctive derivability strategy (as it is carried on in e.g., [9]
following [2] or [10]) is applicable since disjunctive Con (i.e., if $C \vee (A \to B)$,
then $C \vee (\neg B \to \neg A)$) is an admissible rule in $G3_L^1$ since it is admissible in
$G3_L$, that is, the logic containing all and only all $MG3_L$-valid wffs (cf. [4,
§4.3 and also Remark 6.20]). Consequently, we have an adequate Extension
Lemma at our disposal (cf., e.g., [9]), and then the completeness proof can
proceed similarly as in $G3_{\bar{L}}^{\leq}$. However, three points have to be stressed.
(1) The $G3_L^1$-theory $\mathcal{T}$ upon which the canonical $G3_L^1$-model is defined is a
consistent $G3_L^1$-theory. This is immediate since the a-consistency of $\mathcal{T}$ en-
tails its consistency due to its closure under the rule Ecq (cf. Remark 2.7).
(2) The property $R00^*$ holds when interpreted canonically. For suppose for
reductio that there is a wff A such that $A \in \mathcal{T}$ but $A \notin \mathcal{T}^{*^C}$. Then, $\neg A \in \mathcal{T}$
contradicting the consistency of $\mathcal{T}$. (3) (I) in Proposition 4.10 suffices for
the proof of the canonical validity of the conditional clause, condition (iv)
in Definition 3.1.

Based upon the argumentation developed so far in the present section,
we think that we are entitled to state the following theorem.

THEOREM 5.5 (Soundness and completeness of $G3_L^1$). *For any set of wffs*
Γ *and wff* A, $\Gamma \vDash_{2bRG3_L^1} A$ *iff* $\Gamma \vdash_{G3_L^1} A$.

6. A 2 set-up binary Routley semantics for G3

This section on Gödelian 3-valued logic G3 mirrors the preceding section about the logic $G3_L^1$. That is, G3 is endowed with a 2 set-up binary Routley semantics (2bRG3-semantics, for short) w.r.t. which G3 is shown strongly sound and complete.

DEFINITION 6.1 (2bRG3-models). A 2-set-up binary G3-model (2bRG3-model, for short) is a structure $(K, R, *, \vDash)$ where K, R and $\vDash$ are defined exactly as in $2bRG3_L^1$-models but $*$ is a quasi-involutive unary operation on the set K, instead of an involutive one as in Definitions 3.1 and 5.1. That is, we now have: for any $x \in K$, $x^* = x^{**}$.

DEFINITION 6.2 (2bRG3-consequence, 2bRG3-validity). The notions of 2bRG3-consequence and 2bRG3-validity are defined w.r.t. the set K (not only w.r.t. 0 in K) similarly as in $2bRG3_L^\le$-models (and unlike in $2bRG3_L^1$-models).

Regarding Proposition 3.3 and Lemmas 3.4 and 3.5, we note the following facts.

Lemma 3.5 (Entailment Lemma) and conjunction, disjunction and conditional cases in Lemma 3.4 (Hereditary Condition) are proved similarly as in the case of $G3_L^\le$, while the negation case in the latter lemma is proved as follows.

PROPOSITION 6.3 (The negation case in Lemma 3.4). The negation case in Lemma 3.4 holds for G3.

PROOF: (II) A is $\neg B$. (IIa) $a = 0$ and $b = 0^*$. Suppose (1) $R00^*$ and (2) $0 \vDash_M \neg B$ (i.e., $0^* \nvDash_M B$). By quasi-involutiveness of $*$, we get (3) $0^{**} \nvDash_M B$, whence (4) $0^* \vDash_M \neg B$ follows by clause (v) in Definition 3.1. (IIb) $a = 0^*$ and $b = 0$. Suppose (1) $R0^*0$ and (2) $0^* \vDash_M \neg B$. By clause (v) in Definition 3.1, we have (3) $0^{**} \nvDash B$, whence by involutiveness of $*$ (4) $0^* \nvDash_M B$ follows. Finally, (5) $0 \vDash_M \neg B$ is obtained by applying again clause (v) in Definition 3.1. □

Contrary to what the strategy was in the case of $G3_L^\le$, the negation case in Lemma 3.4 has not been proved leaning upon Proposition 3.3, since this proposition only holds from left to right.

PROPOSITION 6.4 ($0^* \vDash \neg A \Rightarrow 0 \nvDash A$). For any 2bRG3-model M and wff A, if $0^* \vDash_{\mathrm{M}} \neg A$, then $0 \nvDash_{\mathrm{M}} A$.

PROOF: Suppose (1) $0^* \vDash_{\mathrm{M}} \neg A$. By clause (v) (Definition 3.1), (2) $0^{**} \nvDash_{\mathrm{M}} A$, whence by quasi-involutiveness of $*$, we get (3) $0^* \nvDash_{\mathrm{M}} A$, and finally, (4) $0 \nvDash_{\mathrm{M}} A$ by Lemma 3.4, $R00^*$ and 3.   □

As regards soundness, the 2bRG3-validity of the contraposition and Efq axioms ($(A \to B) \to (\neg B \to \neg A)$ (A12) and $\neg A \to (A \to B)$ (A13), respectively) is the point of interest, by comparison to $\mathrm{G3}^{\leq}_{\text{Ł}}$ and $\mathrm{G3}^{1}_{\text{Ł}}$, since the rest of the proof proceeds much as the corresponding proofs for the two logics just mentioned. So, let us prove the 2bRG3-validity of A13 as a way of an example.

PROPOSITION 6.5 (Efq is 2bRG3-valid). The Efq axiom $\neg A \to (A \to B)$ (A13) is 2bRG3-valid.

PROOF: A13, $\neg A \to (A \to B)$, is 2bRG3-valid. Suppose that M is a 2bRG3-model falsifying A13. By the Entailment Lemma, for some wffs A, B, either (I) $0 \vDash_{\mathrm{M}} \neg A$ and $0 \nvDash_{\mathrm{M}} A \to B$ or (II) $0^* \vDash_{\mathrm{M}} \neg A$ and $0^* \nvDash_{\mathrm{M}} A \to B$. Case I: We have (1) $0^* \nvDash_{\mathrm{M}} A$ and either (2) $R00$, $0 \vDash_{\mathrm{M}} A$ and $0 \nvDash_{\mathrm{M}} B$ or (3) $R00^*$, $0^* \vDash_{\mathrm{M}} A$ and $0^* \nvDash_{\mathrm{M}} B$. But 3 contradicts 1, whereas (4) $0^* \vDash_{\mathrm{M}} A$ follows from $R00^*$ and 2, contradicting again 1. Case II: we have (1) $0^* \vDash_{\mathrm{M}} \neg A$ (i.e., $0^{**} \nvDash A$) and either (2) $R0^*0^*$, $0^* \vDash_{\mathrm{M}} A$ and $0^* \nvDash_{\mathrm{M}} B$ or (3) $R0^*0$, $0 \vDash_{\mathrm{M}} A$ and $0 \nvDash_{\mathrm{M}} B$. If 2 obtains, by quasi-involutiveness of $*$ and 1, we get (4) $0^* \nvDash_{\mathrm{M}} A$, a contradiction. If (3) is the case, by Proposition 6.4, we have (5) $0^* \nvDash_{\mathrm{M}} \neg A$ contradicting 1.   □

Turning to completeness, the proof can be carried on similarly as that for $\mathrm{G3}^{\leq}_{\text{Ł}}$, given that the sole rule of inference is MP and consequently the disjunctive derivability strategy used in the completeness proof for $\mathrm{G3}^{1}_{\text{Ł}}$ is not needed here. The only worth-remarking differences w.r.t. the completeness proof for $\mathrm{G3}^{\leq}_{\text{Ł}}$ are the following ones: (1) as it was the case with $\mathrm{G3}^{1}_{\text{Ł}}$, (a) the theory $\mathcal{T}$ basing the canonical 2bRG3-model is a consistent 2bRG3-theory. (b) The property $R00^*$ is proved to hold when canonically interpreted by using the consistency of $\mathcal{T}$. (2) $*^C$ is now a quasi-involutive operation on K^C (not an involutive one as in the canonical 2bRG3$^{\leq}_{\text{Ł}}$- and 2bRG3$^{1}_{\text{Ł}}$-models). The fact is proved by using the consistency of $\mathcal{T}$ and the G3-theorem $\neg A \vee \neg\neg A$. (3) As it happened with $\mathrm{G3}^{1}_{\text{Ł}}$, (I) in Proposition 4.10 suffices in order to prove the canonical validity of clause (iv).

The end of section mirrors that of the precedent one.

THEOREM 6.6 (Soundness and completeness of G3). *For any set of wffs* Γ *and wff A,* $\Gamma \vDash_{2bRG3} A$ *iff* $\Gamma \vdash_{G3} A$.

7.   Concluding remarks

In the present paper, a 2 set-up binary Routley semantics (2bR-semantics) is provided for each one of the logics G3, its paraconsistent counterpart, $G3_{\mathrm{L}}^{\leq}$, and an extension of the latter, $G3_{\mathrm{L}}^{1}$. The logics $G3_{\mathrm{L}}^{\leq}$ and $G3_{\mathrm{L}}^{1}$ were introduced in [6], where they were given Hilbert-type axiomatic formulations, once having been interpreted with a 'two-valued' Belnap-Dunn semantics. Recently, Gentzen-type calculi equivalent to the Hilbert-type formulations have been defined in [1].

The different 2bR-semantics defined above have been characterized by having one of the two ensuing features listed in 1, 2 and 3 below.

1. Binary relation R. Property (a) $R00^*$ and property (b) $R00^*$ or $R0^*0$, in addition to reflexivity (i.e,, $R00$ and $R0^*0^*$).

2. Unary relation $*$. (a) Involutiveness. (b) Quasi-involutiveness.

3. Definition of validity. (a) W.r.t. the set K of the two points. (b) Only w.r.t. 0 in K.

But there are other possibilities that may be interesting to examine. For example, inclusion of the property $R0^*0$. Of course, if R is such that both $R00^*$ and $R0^*0$ hold, the resulting 2bR-semantics verifies all classical tautologies. But what about $R0^*0$ and involutiveness? Or what about $R0^*0$ and quasi-involutiveness? And which is the notion of validity the 2bR-semantics is going to be defined with? Are there interesting systems characterized by the sketched 2bR-semantics?

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