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A NOTE ON GÖDEL-DUMMET LOGIC LC

Abstract

Let $A_0, A_1, ..., A_n$ be (possibly) distintict wffs, n being an odd number equal to or greater than 1. Intuitionistic Propositional Logic IPC plus the axiom $(A_0 \to A_1) \vee ... \vee (A_{n-1} \to A_n) \vee (A_n \to A_0)$ is equivalent to Gödel-Dummett logic LC. However, if n is an even number equal to or greater than 2, IPC plus the said axiom is a sublogic of LC.

Keywords: Intermediate logics, Gödel-Dummett logic LC.

1. Introduction

Propositional Intuitionistic Logic IPC can be axiomatized as follows (cf. [5] and references therein):

Axioms:

A1. $A \to (B \to A)$

A2. $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$

A3. $(A \wedge B) \to A$; $(A \wedge B) \to B$

A4. $A \to [B \to (A \wedge B)]$

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A5. $A \to (A \vee B)$; $B \to (A \vee B)$

A6. $(A \vee B) \to [(A \to C) \to [(B \to C) \to C]]$

A7. $(A \to \neg B) \to (B \to \neg A)$

A8. $\neg A \to (A \to B)$

Rule of inference:

Modus Ponens (MP): If A and $A \to B$, then B

The following wffs and rule (derivable in IPC) are used in the sequel:

t1. $A \to A$

t2. $(B \to C) \to [(A \to B) \to (A \to C)]$

t3. $(A \to B) \to [(B \to C) \to (A \to C)]$

Transitivity (Trans): If $A \to B$ and $B \to C$, then $A \to C$

In what follows, regardless of a particular order or association of the n implicative wffs $A_1, ..., A_n$ connected by $\vee$ as the sole connective, in general, we simply write $A_1 \vee A_2 ... \vee A_n$.

By IPC$_+$, we refer to the negationless fragment of IPC, axiomatized by A1 through A6 and MP. Well then, in [4] it is noted that Gödel-Dummett logic LC (cf. [2], [3]) can be axiomatized by adding any of the following axiom schemes to IPC:

a1. $(A \to B) \vee (B \to A)$

a2. $(A \to B) \vee [(A \to B) \to A]$

a3. $(A \to B) \vee [(A \to B) \to B]$

a4. $[A \to (B \vee C)] \to [(A \to B) \vee (A \to C)]$

a5. $[(A \wedge B) \to C] \to [(A \to C) \vee (B \to C)]$

a6. $[[(A \to B) \to B] \wedge [(B \to A) \to A]] \to (A \vee B)$

We remark that Dummett's original axiomatization of LC is the result of adding a1 to IPC (cf. [2]). We will occasionally refer to a1 as "Dummett's axiom".

The authors of [4] add: "An even larger number of equivalents [axioms] arises by the fact that in IPC $\vdash A \vee B$ iff $\vdash (A \to B) \wedge (B \to C) \to C$ (**DR**), and, more generally, $\vdash D \to A \vee B$ iff $\vdash D \wedge (A \to C) \wedge (B \to C) \to C$ (**EDR**)" ([2], p. 1).

The aim of this note is to increase the number of equivalent axioms recorded above by showing that, for any odd number n equal to or greater than 1 and (possibly) distinct wffs $A_1, A_2, ..., A_n$, addition of

$$A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0$$

to IPC is an axiomatization of LC.

As a by-product of the fact just stated, it also will be shown that if in the preceding wff n is an even number equal to or greater than 2, addition of it to IPC results in an intermediate logic included in (but not including) LC.

To the best of our knowledge, neither of these facts is recorded in the literature.

2. IPC plus $(A \to B) \vee [(B \to C) \vee (C \to A)]$

Let $A_0, A_1, ...A_n, A_{n+1}, A_{n+2}$ be (possibly) distinct wffs, n being an even number equal to or greater than 2. Consider now the following wffs:

α. $A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_0$

β. $A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0$

γ. $A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_{n+1} \vee A_{n+1} \to A_{n+2} \vee A_{n+2} \to A_0$

We prove:

Proposition 2.1 (IPC$_+$ & β proves α). The wff α is provable in IPC$_+$ plus β.

Proof:

$$1.\ A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0 \qquad \beta$$

By changing in (1), for each $i \geq 3$, A_i by A_1 (resp., A_2) if i is an odd number (resp., even number), we get

$$2.\ A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_1 \vee A_2 \to A_0$$

or equivalently

$$3.\ A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_0 \vee A_2 \to A_1$$

Moreover, by changing in (1), for each $i \geq 3$, A_i by A_0 (resp., A_1) if i is an odd number (resp., even number), we get

$$4.\ A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_0 \vee A_1 \to A_0$$

Next, we proceed as follows. Obviously, we have

$$5.\ (A_2 \to A_0) \to (\alpha)$$

In addition,

6. $(A_1 \to A_0) \to [(A_2 \to A_1) \to (A_2 \to A_0)]$	t2
7. $(A_1 \to A_0) \to [(A_2 \to A_1) \to (\alpha)]$	t2, Trans, 5, 6
8. $(\alpha) \to [(A_2 \to A_1) \to (\alpha)]$	A1

Then,

9. $(A_2 \to A_1) \to (\alpha)$	A6, 4, 7, 8

Now, by using

10. $(\alpha) \to (\alpha)$	t1

3, 9 and A6, we derive

$$11.\ A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_0$$

as it was to be proved. □

Proposition 2.2 (IPC$_+$ & α proves β). The wff β is provable in IPC$_+$ plus α.

Proof: Firstly, we show,

(I) The wff δ, $A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_3 \vee A_3 \to A_4 \vee A_4 \to A_0$, is provable in IPC$_+$ plus α:

1. $A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_0$	α
2. $A_2 \to A_3 \vee A_3 \to A_4 \vee A_4 \to A_2$	α

We trivially have:

$$3.\ (A_0 \to A_1 \vee A_1 \to A_2) \to (\delta)$$
$$4.\ (A_2 \to A_3 \vee A_3 \to A_4) \to (\delta)$$
$$5.\ (A_4 \to A_0) \to (\delta)$$

Then, we get

$$6.\ [(A_4 \to A_2) \to (\delta)] \to (\delta) \qquad \text{A6, 2, 4}$$

In addition,

$$7.\ (A_2 \to A_0) \to [(A_4 \to A_2) \to (A_4 \to A_0)] \qquad \text{t2}$$
$$8.\ (A_2 \to A_0) \to [(A_4 \to A_2) \to (\delta)] \qquad \text{t2, Trans, 5, 7}$$
$$9.\ (A_2 \to A_0) \to (\delta) \qquad \text{Trans, 6, 8}$$

Finally,

$$10.\ A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_3 \vee A_3 \to A_4 \vee A_4 \to A_0 \qquad \text{A6, 1, 3, 9}$$

(II) Given (I), the wff ε, $A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_3 \vee A_3 \to A_4 \vee A_4 \to A_5 \vee A_5 \to A_6 \vee A_6 \to A_0$, is provable in IPC$_+$ plus α similarly as δ has been proved above. We can use δ, α and t2 in the forms $A_4 \to A_5 \vee A_5 \to A_6 \vee A_6 \to A_4$ and $(A_4 \to A_0) \to [(A_6 \to A_4) \to (A_6 \to A_0)]$, respectively.

(III) In this way, the wff γ, displayed at the beginning of the section, can be obtained given β (i.e., $A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n$), and α and t2 in the forms $A_n \to A_{n+1} \vee A_{n+1} \to A_{n+2} \vee A_{n+2} \to A_n$ and $(A_n \to A_0) \to [(A_{n+2} \to A_n) \to (A_{n+2} \to A_0)]$, respectively.

Once (I), (II) and (III) are proved, it is clear that β is derivable from IPC$_+$ plus α. □

Given Propositions 2.1 and 2.2, we have the following corollary.

Corollary 2.3 (IPC & α is equivalent to IPC & β). Let $A_0, A_1, ..., A_n$ be (possibly) distinct wffs, n being an even number equivalent to or greater than 2. The systems IPC plus α (i.e., $A_0 \to A_1 \vee A_1 \to A_2 \vee A_2 \to A_0$) and IPC plus β (i.e., $A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0$) are deductively equivalent.

The section is ended by proving that Dummett's axiom $(A \to B) \vee (B \to A)$ (a1) is not provable from IPC plus $(A \to B) \vee [(B \to C) \vee (C \to A)]$. Let us provisionally name LC_2 the result of adding $(A \to B) \vee [(B \to C) \vee (C \to A)]$ to IPC. We have:

Proposition 2.4 (Dummett's axiom is not provable in LC_2). Dummett's axiom $(A \to B) \vee (B \to A)$ is not provable in LC_2, that is, the result of adding $(A \to B) \vee [(B \to C) \vee (C \to A)]$ to IPC.

Proof: Consider the following set of truth-tables (4 is the only designated value):

$\to$	0	1	2	3	4	$\neg$
0	4	4	4	4	4	4
1	2	4	2	4	4	2
2	1	1	4	4	4	1
3	0	1	2	4	4	0
4	0	1	2	3	4	0

$\wedge$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	1	1
2	0	0	2	2	2
3	0	1	2	3	3
4	0	1	2	3	4

$\vee$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	3	3	4
2	2	3	2	3	4
3	3	3	3	3	4
4	4	4	4	4	4

This set verifies all axioms of IPC (A1-A8) plus $(A \to B) \vee [(B \to C) \vee (C \to A)]$ and the rule MP, but falsifies Dummett's axiom: let v be any assignment to the propositional variables such that $v(p) = 2$ and $v(q) = 1$, for distinct propositional variables p and q. Then, $v[(p \to q) \vee (q \to p)]= 3$. □

It follows from this proposition that LC is not included in LC_2. Instead, in the following section, it is proved that LC_2 is included in LC.

3. A sequence of axioms equivalent to Dummett's axiom

Let $A_0, A_1, ..., A_n$ be distinct wffs, n being an odd number equal to or greater than 1. Now, consider the following wffs:

ε. $A_0 \to A_1 \vee A_1 \to A_0$

θ. $A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0$

We prove:

PROPOSITION 3.1 (IPC$_+$ & θ proves ε). The wff ε is provable from IPC$_+$ plus θ.

PROOF:

$$1.\ A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0 \qquad \theta$$

By changing in (1), for each $i \geq 2$, A_i by A_0 (resp., A_1) if i is an even number (resp., odd number), we get

$$2.\ A_0 \to A_1 \vee A_1 \to A_0 \vee ... \vee A_0 \to A_1 \vee A_1 \to A_0$$

that is,

$$3.\ A_0 \to A_1 \vee A_1 \to A_0$$

i.e., the characteristic axiom of LC. □

PROPOSITION 3.2. Consider the following wff η, $A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0$, where $A_0, A_1, ..., A_{n-1}, A_n$ are (possibly) distinct wffs. This wff η is provable in LC (notice that n is any natural number equal to or greater than 1).

PROOF:

1. $(A_n \to A_{n-1}) \to [(A_{n-1} \to A_{n-2}) \to (A_n \to A_{n-2})]$ t3
2. $(A_n \to A_{n-2}) \to [(A_{n-2} \to A_{n-3}) \to (A_n \to A_{n-3})]$ t3
3. $(A_n \to A_{n-1}) \to [(A_{n-1} \to A_{n-2}) \to [(A_{n-2} \to A_{n-3}) \to (A_n \to A_{n-3})]]$ t2, Trans, 1, 2

In this way, we have

4. $(A_n \to A_{n-1}) \to [(A_{n-1} \to A_{n-2}) \to [... \to [(A_1 \to A_0) \to (A_n \to A_0)]...]]$

Now, we obviously have

$$5.\ (A_n \to A_0) \to (\eta)$$

and

$$6.\ (A_{n-1} \to A_n) \to (\eta)$$

So, by t2, t3, (4) and (5), we derive

$$7.\ (A_n \to A_{n-1}) \to [(A_{n-1} \to A_{n-2}) \to [... \to [(A_1 \to A_0) \to (\eta)]...]]$$

And by A1, (6) and Trans, we obtain

$$8.\ (A_{n-1} \to A_n) \to [(A_{n-1} \to A_{n-2}) \to [... \to [(A_1 \to A_0) \to (\eta)]...]]$$

Now, by Dummett's axiom, we have

$$9.\ (A_{n-1} \to A_n) \vee (A_n \to A_{n-1})$$

whence

$$10.\ (A_{n-1} \to A_{n-2}) \to [(A_{n-2} \to A_{n-3}) \to [... \to [(A_1 \to A_0) \to (\eta)]...]]$$

follows by A6, (7), (8) and (9).

Next, notice that, for any k $(0 \leq k \leq n-1)$,

$$11.\ (A_k \to A_{k+1}) \to (\eta)$$

is clearly provable.

Finally, proceeding from (10) and (11), similarly as we have proceeded from (4), (7), (8) and (9) to (10), we eventually derive

$$12.\ A_0 \to A_1 \vee ...\ A_{n-1} \to A_n \vee A_n \to A_0$$

that is, the wff η, as it was to be proved. □

Given Propositions 3.1 and 3.2, we immediately have the following corollary.

Corollary 3.3 (IPC & θ is equivalent to LC). Let $A_0, A_1, ..., A_n$ be (possibly) distinct wffs, n being an odd number equivalent to or greater than 1. The result of adding the wff θ (i.e., $A_0 \to A_1 \vee ... A_{n-1} \to A_n \vee A_n \to A_0$) to IPC is a system deductively equivalent to LC.

On the other hand, given Propositions 2.4 and 3.2, the following corollary is immediate.

COROLLARY 3.4 (LC_2 is included in LC). The system LC_2, that is, IPC plus the axiom $(A \to B) \vee [(B \to C) \vee (C \to A)]$ is included in (but does not include) LC.

4. A couple of remarks

This note is ended with a couple of remarks.

1. The proofs of Propositions 2.1, 2.2, 3.1 and 3.2 are given within the context of IPC_+, but it is possible that weaker systems are sufficient. For example, MaGIC (cf. [7]) does not find a set of truth-tables verifying *Ticket Entailment* (cf. [1]) plus Dummett's axiom but falsifying $(A \to B) \vee (B \to C) \vee (C \to D) \vee (D \to A)$.

2. An IPC-model is the following structure $(K, R, \vDash)$, where K is a non-empty set, R is a reflexive and transitive binary relation defined on K and $\vDash$ is a (valuation) relation such that for each $a \in K$, propositional variable p and wffs A, B, the following conditions (clauses) are fulfilled:

(i) $(Rab \ \& \ a \vDash p) \Rightarrow b \vDash p$

(ii) $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$

(iii) $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$

(iv) $a \vDash A \to B$ iff for all $b \in K$, $(Rab$ and $b \vDash A) \Rightarrow b \vDash B$

(v) $a \vDash \neg A$ iff for all $b \in K$, $Rab \Rightarrow b \nvDash A$

We have: for any set of wffs Γ and wff A, $\Gamma \vdash_{IPC} A$ iff $\Gamma \vDash A$ ($\Gamma \vDash A$ iff for any IPC-model $\mathcal{M}$ and $a \in K$, $a \vDash A$ if $a \vDash \Gamma$, where $a \vDash \Gamma$ iff $a \vDash B$ for all $B \in \Gamma$) (cf. [5] or [6] and references therein).

Well then, let us name LC_n the result of adding the axiom

$$A_0 \to A_1 \vee ... \vee A_{n-1} \to A_n \vee A_n \to A_0$$

to IPC; and let LC_n-models be the result of adding the following condition to IPC-models: for any $a_0, a_1, ..., a_n \in K$, if Ra_0a_1 and Ra_0a_2 and ,..., and Ra_0a_n, then, Ra_1a_n or Ra_2a_1 or ,..., or Ra_na_{n-1}. For instance, LC_2-models (i.e., models for IPC plus the axiom $(A \to B) \vee (B \to C) \vee (C \to A)$)

are defined when adding to IPC-models the condition, for any $a, b, c, d \in K$, $(Rab\ \&\ Rac\ \&\ Rad) \Rightarrow (Rbd$ or Rcb or $Rdc)$. It is not difficult to prove that LC_n is (strongly) sound and complete w.r.t. LC_n-models.

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References

[1] A. R. Anderson, N. D. Belnap Jr., **Entailment. The Logic of Relevance and Necessity**, vol. I, Princeton University Press, Princeton, NJ (1975).

[2] M. Dummett, *A propositional calculus with denumerable matrix*, **Journal of Symbolic Logic**, vol. 24(2) (1959), pp. 97–106, DOI: https://doi.org/10.2307/2964753.

[3] K. Gödel, *Zum intuitionistischen Aussagenkalkül*, **Anzeiger der Akademie der Wissenschaften in Wien**, vol. 69 (1932), pp. 65–66.

[4] D. D. Jongh, F. S. Maleki, *Below Gödel-Dummett*, [in:] **Booklet of abstracts of Syntax meets Semantics 2019 (SYSMICS 2019)**, Institute of Logic, Language and Computation, University of Amsterdam (2019), pp. 99–102.

[5] J. Moschovakis, *Intuitionistic Logic*, [in:] E. N. Zalta (ed.), **The Stanford Encyclopedia of Philosophy**, winter 2018 ed. (2018), URL: https://plato.stanford.edu/archives/win2018/entries/logic-intuitionistic.

[6] G. Robles, J. M. Méndez, *A binary Routley semantics for intuitionistic De Morgan minimal logic HM and its extensions*, **Logic Journal of the IGPL**, vol. 23(2) (2014), pp. 174–193, DOI: https://doi.org/10.1093/jigpal/jzu029.

[7] J. K. Slaney, *MaGIC, Matrix Generator for Implication Connectives: Version 2.1, Notes and Guide* (1995), http://users.cecs.anu.edu.au/jks/magic.html.

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