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ON THE CATEGORY OF EQ -ALGEBRAS

Abstract

In this paper, we studied the category of EQ -algebras and showed that it is complete, but it is not cocomplete, in general. We proved that multiplicatively relative EQ -algebras have coequalizers and we calculated coproduct and pushout in a special case. Also, we constructed a free EQ -algebra on a singleton.

Keywords: EQ -algebras, free EQ -algebras, category theory, universal algebra, variety.

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1. Introduction

Fuzzy type theory was developed as a counterpart of the classical higher-order logic. Since the algebra of truth values is no longer a residuated lattice, a specific algebra called an EQ -algebra was proposed by Novák [16, 17, 18]. The main primitive operations of EQ -algebras are meet, multiplication, and fuzzy equality. Implication is derived from fuzzy equality and it is not a residuation with respect to multiplication. Consequently, EQ -algebras overlap with residuated lattices but are not identical to them. Novák and De Baets in [18] introduced various kinds of EQ -algebras.

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Novák and El-Zekey in [14], proved that the class of EQ -algebras is a variety. In [19], J. Yang and X. Zhang introduced a new class of EQ -algebras, i.e., multiplicatively relative EQ -algebras. Also, they defined the notion of a filter generated by a nonempty subset.

Category theory is a powerful language, or conceptual framework, allowing us to see the universal components of a family of structures of a given kind, and how structures of different kinds are interrelated. Category theory is both an interesting object of philosophical study, and a potentially powerful formal tool for philosophical investigations of concepts such as space, system, and even truth. In [1], it has shown that the variety algebras, together with its homomorphisms, form a category and also, every non-trivial variety of algebras contains a free object on a given set. The category of some algebraic structures are studied. It is well known that the category of groups, rings, modules, and vector spaces are complete and cocomplete. The category of some logical algebraic structures have been studied well, too. For example, it has been proved that the category of Boolean algebras is isomorphic to the subcategory of rings named as Boolean rings [8]. Also, it is known that the category of MV -algebras is equivalent to that of unital lattice ordered groups (ℓ -groups). This equivalence, which depends in large part on the natural algebraic addition of MV -algebras [9] has been an essential tool in the study of MV -algebras. The categories of some other algebraic structures such as BCK(BCI)-algebras, BL-algebras, soft and rough sets have been studied, too (see [5, 6, 7, 10, 11, 12]).

With these inspirations, we studied the category of EQ -algebras and showed that it is complete, but it is not cocomplete, in general. We proved that multiplicatively relative EQ -algebras have coequalizers and we calculated coproduct and pushout in a special case. Also, we constructed a free EQ -algebra on a singleton.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in this paper (See [13, 14, 19]).

An EQ -algebra is an algebraic structure $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$, where for any $a, b, c, d \in E$, the following statements hold:

- (E1) $(E, \wedge, 1)$ is a \wedge -semilattice with top element 1. For any $a, b \in E$, we set $a \leq b$ if and only if $a \wedge b = a$.

(E2) $(E, \otimes, 1)$ is a (commutative) monoid and \otimes is isotone with respect to \leq .

(E3) $a \sim a = 1$.

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq (c \sim (d \wedge b))$.

(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$.

(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$.

(E7) $a \otimes b \leq a \sim b$.

The operations " \wedge ", " \otimes ", and " \sim " are called *meet*, *multiplication*, and *fuzzy equality*, respectively. For any $a, b \in E$, we defined the binary operation *implication* on E by, $a \rightarrow b = (a \wedge b) \sim a$. Also, in particular $1 \rightarrow a = 1 \sim a = \tilde{a}$.

Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra and $a, b, c, d \in E$ are arbitrary elements. Then \mathcal{E} is called *separated*, if $a \sim b = 1$, then $a = b$, *good*, if $\tilde{a} = a$, *residuated*, where $(a \otimes b) \wedge c = a \otimes b$ if and only if $a \wedge ((b \wedge c) \sim b) = a$, *lattice-ordered EQ-algebra*, if it has a lattice reduct,¹ *lattice EQ-algebra* (or *lEQ-algebra*), if it is a lattice-ordered EQ-algebra and

$$((a \vee b) \sim c) \otimes (d \sim a) \leq ((d \vee b) \sim c),$$

multiplicatively relative EQ-algebra, if $a \sim b \leq (a \otimes c) \sim (b \otimes c)$.

PROPOSITION 2.1 ([14]). Let \mathcal{E} be an EQ-algebra. Then for any $a, b, c \in E$, \mathcal{E} is residuated if and only if \mathcal{E} is good and $(a \otimes b) \rightarrow c \leq a \rightarrow (b \rightarrow c)$.

PROPOSITION 2.2 ([19]). Each linear and residuated EQ-algebra is multiplicatively relative.

PROPOSITION 2.3 ([14]). Let \mathcal{E} be an EQ-algebra. Then, for all $a, b, c \in E$ (i) $a \sim b = b \sim a$ and (ii) $a \otimes (a \sim b) = \tilde{b}$.

Let $\mathcal{E} = (E, \otimes_{\mathcal{E}}, \wedge_{\mathcal{E}}, \sim_{\mathcal{E}}, 1_{\mathcal{E}})$ and $\mathcal{G} = (G, \otimes_{\mathcal{G}}, \wedge_{\mathcal{G}}, \sim_{\mathcal{G}}, 1_{\mathcal{G}})$ be two EQ-algebras. A map $f : E \rightarrow G$ is an EQ-homomorphism, if for any $a, b \in E$, $f(a \otimes_{\mathcal{E}} b) = f(a) \otimes_{\mathcal{G}} f(b)$, $f(a \wedge_{\mathcal{E}} b) = f(a) \wedge_{\mathcal{G}} f(b)$, $f(a \sim_{\mathcal{E}} b) = f(a) \sim_{\mathcal{G}} f(b)$,

¹Given an algebra $\langle E, F \rangle$, where F is a set of operations on E and $F' \subseteq F$, then the algebra $\langle E, F' \rangle$ is called the F' -reduct of $\langle E, F \rangle$.

and $f(1_{\mathcal{E}}) = 1_{\mathcal{G}}$. A nonempty subset F of \mathcal{E} is called a filter of \mathcal{E} if for any $a, b, c \in E$: (F1) if $a \in F$ and $a \leq b$, then $b \in F$, (F2) if $a, b \in F$, then $a \otimes b \in F$, (F3) if $a \sim b \in F$, then $(a \otimes c) \sim (b \otimes c) \in F$.

PROPOSITION 2.4 ([19]). Let $f : \mathcal{E} \rightarrow \mathcal{G}$ be an EQ -homomorphism. If \mathcal{G} is separated, then $\ker(f) = \{a \in E | f(a) = 1\}$ is a filter of \mathcal{E} .

PROPOSITION 2.5 ([19]).

- (i) Let $\{F_i | i \in I\}$ be a family of filters of an EQ -algebra \mathcal{E} . Then $\bigcap_{i \in I} F_i$ is a filter of \mathcal{E} .
- (ii) Let $\{\mathcal{E}_i | i \in I\}$ be a family of EQ -algebras and F_i be a filter of E_i for any $i \in I$. Then $F = \prod F_i$ is a filter of $\mathcal{E} = \prod \mathcal{E}_i$.

THEOREM 2.6 ([14]). Let F be a filter of EQ -algebra \mathcal{E} . A binary relation \approx_F on E which is defined by $a \approx_F b$ if and only if $a \sim b \in F$, is a congruence relation on \mathcal{E} and $\mathcal{E}/F = (E/F, \wedge_F, \otimes_F, \sim_F, F)$ is a separated EQ -algebra, where, for any $a, b \in E$, we have,

$$[a] \wedge_F [b] = [a \wedge b], \quad [a] \otimes_F [b] = [a \otimes b], \quad [a] \sim_F [b] = [a \sim b].$$

A binary relation \leq_F on E/F which is defined by $[a] \leq_F [b]$ if and only if $[a] \wedge_F [b] = [a]$ is a partial order on E/F and for any $[a], [b] \in \mathcal{E}/F$, $[a] \leq_F [b]$ if and only if $a \rightarrow b \in F$.

THEOREM 2.7. [19] Let X be a nonempty subset of a multiplicatively relative EQ -algebra \mathcal{E} . Then

$$\langle X \rangle = \{a \in E \mid \exists n \in \mathbb{Z}^+, x_i \in X \text{ s.t. } x_1 \otimes x_2 \otimes \cdots \otimes x_n \leq a\}$$

is a generated filter by X .

Now, we present some definitions and results in category theory which will be used in this paper (see [1, 2, 3, 8, 15]). A *category* consists of *objects*: A, B, X, \dots and *morphisms* (arrows): f, g, h, \dots . For each morphism f , there are given objects $\text{dom}(f), \text{cod}(f)$ called the *domain* and *codomain* of f . We write $f : A \rightarrow B$ to indicate that $A = \text{dom}(f)$ and $B = \text{cod}(f)$. Given morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$, where $\text{cod}(f) = \text{dom}(g)$ there is given a morphism $g \circ f : A \rightarrow C$ called the composite of f and g . For each object A , there is given a morphism $\text{id}_A : A \rightarrow A$ called the *identity* morphism of A . These data are required to satisfy the laws, for

all $f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D, h \circ (g \circ f) = (h \circ g) \circ f$ and $f \circ id_A = f = id_B \circ f$.

A category \mathcal{C} is called *locally small* if for all objects X, Y in \mathcal{C} , the collection $Hom_{\mathcal{C}}(X, Y) = \{f \in \mathcal{C} \mid f : X \rightarrow Y\}$ is a set (called a hom-set). A category is said to be *small* if its objects form a set. A category \mathcal{C} is called *connected*, if for all objects X, Y in $\mathcal{C}, Hom_{\mathcal{C}}(X, Y) \neq \emptyset$. A *functor* F from a category \mathcal{C} to a category \mathcal{D} is a prescription that assigns to every object A of \mathcal{C} an object $F(A)$ of \mathcal{D} , and to every morphism $\alpha : A \rightarrow B$ of \mathcal{C} a morphism $F(\alpha) : F(A) \rightarrow F(B)$ of \mathcal{D} , such that (i): $F(id_A) = id_{F(A)}$, for every object A of \mathcal{C} and (ii): if $\beta \circ \alpha$ is defined in \mathcal{C} , then $F(\beta) \circ F(\alpha)$ is defined in \mathcal{D} and $F(\beta) \circ F(\alpha) = F(\beta \circ \alpha)$. A *monomorphism* (also called a *monic* morphism) is a left-cancellative morphism. That is, a morphism $f : X \rightarrow Y$ such that for all objects Z and all morphisms $g_1, g_2 : Z \rightarrow X$, if $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$. An *epimorphism* (also called an *epic* morphism) is a morphism $f : X \rightarrow Y$ that is right-cancellative in the sense that, for all objects Z and all morphisms $g_1, g_2 : Y \rightarrow Z$, if $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$. An *initial object* is an object I such that for every object X , there exists precisely one morphism $I \rightarrow X$. A *terminal object* is an object T such that for every object X , there exists precisely one morphism $X \rightarrow T$. If an object is both initial and terminal, it is called a *zero object*. An object Q in a category \mathcal{C} is said to be *injective* if for every monomorphism $f : X \rightarrow Y$ and every morphism $g : X \rightarrow Q$ there exists a morphism $h : Y \rightarrow Q$ such that $h \circ f = g$. If B is an object of a category \mathcal{C} , then by a *subobject* of B we mean a pair of (A, f) consisting of an object A of \mathcal{C} and a morphism $f : A \rightarrow B$ that is monic. Let X be a set (called a basis), A be an object, and $i : X \rightarrow A$ be an injective map between sets (called the canonical insertion). We say that A is the *free object* on X (with respect to i) if and only if it satisfies the following universal property: for any object B and any map between sets $f : X \rightarrow B$, there exists a unique morphism $g : A \rightarrow B$ such that $f = g \circ i$. Let A_1 and A_2 be two algebras of the same type F . The *product* $A_1 \times A_2$ is an algebraic structure whose universe is the set $A_1 \times A_2$, such that for $f \in F_n$ and $a_i \in A_1, a'_i \in A_2, 1 \leq i \leq n$,

$$f^{A_1 \times A_2}(\langle a_1, a'_1 \rangle, \dots, \langle a_n, a'_n \rangle) = \langle f^{A_1}(a_1, \dots, a_n), f^{A_2}(a'_1, \dots, a'_n) \rangle.$$

Let A and B be two objects and $f, g : A \rightarrow B$ be two morphisms. The *equalizer* of f and g is an object E and a morphism $eq : E \rightarrow A$ satisfying $f \circ eq = g \circ eq$ such that, for any object O and morphism $m : O \rightarrow A$,

if $f \circ m = g \circ m$, then there exists a unique morphism $u : O \rightarrow E$ such that $eq \circ u = m$. Let A be an object and $f : Y \rightarrow A$ and $g : Z \rightarrow A$ be two morphisms. The *pullback* of f, g is an object P with two morphisms $p_1 : P \rightarrow Y$ and $p_2 : P \rightarrow Z$ such that $f \circ p_1 = g \circ p_2$ and for any $q_1 : Q \rightarrow Y$ and $q_2 : Q \rightarrow Z$ with $f \circ q_1 = g \circ q_2$, there exists a unique $u : Q \rightarrow P$ such that $q_1 = p_1 \circ u$ and $q_2 = p_2 \circ u$. Let \mathcal{J} and \mathcal{C} be categories. A *diagram* of type \mathcal{J} in \mathcal{C} is a functor $D : \mathcal{J} \rightarrow \mathcal{C}$. We will write the objects in the index category \mathcal{J} lower case, i, j, \dots and the values of the functor $D : \mathcal{J} \rightarrow \mathcal{C}$ in the form D_i, D_j , etc. If \mathcal{J} is a small category, then $D : \mathcal{J} \rightarrow \mathcal{C}$ is a *small diagram*. A *cone* to a diagram D consists of an object C in \mathcal{C} and a family of morphisms in \mathcal{C} , $c_j : C \rightarrow D_j$ one for each object $j \in \mathcal{J}$, such that for each morphism $\alpha : i \rightarrow j$ in \mathcal{J} , such that $D_\alpha \circ c_i = c_j$. A *morphism of cones* $\vartheta : (C, c_j) \rightarrow (C', c'_j)$ is a morphism ϑ in \mathcal{C} such that for any $j \in \mathcal{J}$, $c_j = c'_j \circ \vartheta$. Thus, we have an apparent category $\mathbf{Cone}(D)$ of cones to D . A *limit* for a diagram $D : \mathcal{J} \rightarrow \mathcal{C}$ is a terminal object in $\mathbf{Cone}(D)$. A category \mathcal{C} is called *small-complete* if all small diagrams in \mathcal{C} have limits in \mathcal{C} .

Suppose that $(A_i)_{i \in I}$ is a family of subobjects of a given object B . Constructing a category \mathcal{K} as follows: for the objects of \mathcal{K} take those subobjects (D, d) of B for which there exists a commutative triangle as follows.

$$\begin{array}{ccc} D & \xrightarrow{d_i} & A_i \\ & \searrow d & \downarrow f_i \\ & & B \end{array}$$

For the morphisms from (D, d) to (E, e) take those morphisms $\alpha : D \rightarrow E$ in \mathcal{C} such that the following diagram is commutative.

$$\begin{array}{ccc} D & & \\ \downarrow \alpha & \searrow d & \\ E & \xrightarrow{e} & B \end{array}$$

The terminal object in \mathcal{K} is called an *intersection* of the family $(A_i, f_i)_{i \in I}$ of subobjects of B .

PROPOSITION 2.8 ([3]). A category \mathcal{C} has finite products if and only if it has a terminal object and every pair of objects in \mathcal{C} has a product.

THEOREM 2.9 ([2, 3]). *Let \mathcal{C} be a category. The following statements are equivalent:*

- (i) \mathcal{C} has finite products and equalizers.
- (ii) \mathcal{C} has finite products and finite intersections.
- (iii) \mathcal{C} has pullbacks and a terminal object.
- (iv) \mathcal{C} has all finite limits.

Let A and B be two objects and $f, g : A \rightarrow B$ be two morphisms. The *coequalizer* of f and g is an object Q and a morphism $q : B \rightarrow Q$ such that $q \circ f = q \circ g$. Moreover, for any object Q' and morphism $q' : B \rightarrow Q'$ there exists a unique morphism $u' : Q \rightarrow Q'$ such that $u' \circ q = q'$. Let X be an object and $f : X \rightarrow Y$ and $g : X \rightarrow Z$ be two morphisms. The *pushout* of f, g is an object P with two morphisms $i_1 : Y \rightarrow P$ and $i_2 : Z \rightarrow P$ such that $i_1 \circ f = i_2 \circ g$ and for any $q_1 : Y \rightarrow Q$ and $q_2 : Z \rightarrow Q$ with $q_1 \circ f = q_2 \circ g$, there exists a unique $u : P \rightarrow Q$ such that $q_1 = u \circ i_1$ and $q_2 = u \circ i_2$.

THEOREM 2.10 ([8]). *A nonempty class K of algebraic structures of type F is called a variety if it is closed under subalgebras, homomorphic images, and direct products.*

THEOREM 2.11 ([8]). *Every variety has free objects.*

Notation. From now on, in this paper, $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ or simply \mathcal{E} is an EQ -algebra, unless otherwise state.

3. Category of EQ -algebras

If we consider EQ -algebras and EQ -homomorphisms between them as objects and morphisms, then class of all EQ -algebras and the EQ -homomorphisms with the usual composition of maps forms a locally small category which is denoted by \mathcal{EQ} . In the rest of this article, we study the morphisms, objects, limits and colimits of \mathcal{EQ} .

3.1. Morphisms

In this subsection, we give the conditions that when an injective EQ -homomorphism is monic. Also, we show that an onto EQ -homomorphism is epic but the converse is not true, in general.

PROPOSITION 3.1. $\mathcal{E}\mathcal{Q}$ is connected.

PROOF: Let \mathcal{E} and \mathcal{G} be two EQ -algebras. Then the map $e : \mathcal{E} \rightarrow \mathcal{G}$, where for any $a \in E$, $e(a) = 1_{\mathcal{G}}$ is a homomorphism. Thus, $Hom(\mathcal{E}, \mathcal{G}) \neq \emptyset$. \square

PROPOSITION 3.2. Let $f : \mathcal{E} \rightarrow \mathcal{G}$ and $ker f = \{a \in E | f(a) = 1_{\mathcal{G}}\} = \{1_{\mathcal{E}}\}$. If \mathcal{E} is separated, then f is injective.

PROOF: Suppose that for some $a, b \in E$, $f(a) = f(b)$. Since f is a homomorphism, by (E3) we have, $1_{\mathcal{G}} = f(a) \sim f(b) = f(a \sim b)$. Then $a \sim b \in ker f$ and so $a \sim b = 1$. Since \mathcal{E} is separated, we have $a = b$ and f is injective. \square

In the following example, we show that the separated condition in Proposition 3.2 is necessary.

Example 3.3. Let $H = \{0, a, b, 1\}$ be a chain where $0 \leq a \leq b \leq 1$. For any $x, y \in H$, we define the operations \otimes and \sim on H as Table 1 and Table 2:

Table 1

\otimes	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

Table 2

\sim	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	1
1	0	a	1	1

Then $\mathcal{H} = (H, \wedge, \otimes, \sim, 1)$ is a non-separated EQ -algebra. Let $G = \{0, d, 1\}$ be a chain where $0 \leq d \leq 1$. For any $x, y \in G$, we define the operations \otimes and \sim on G as Table 3 and Table 4:

Table 3

\otimes	0	d	1
0	0	0	0
d	0	0	d
1	0	d	1

Table 4

\sim	0	d	1
0	1	0	0
d	0	1	1
1	0	1	1

Then $\mathcal{G} = (G, \wedge, \otimes, \sim, 1)$ is an EQ-algebra and $f : H \rightarrow G$ is an EQ-homomorphism where $f(0) = f(a) = 0$, $f(b) = d$ and $f(1) = 1$. Thus, it is clear that $\ker f = \{1\}$ but f is not injective.

PROPOSITION 3.4. Let $f : \mathcal{E} \rightarrow \mathcal{G}$. If f is injective, then $\ker f = \{1_{\mathcal{E}}\}$.

THEOREM 3.5. Let $f \in \text{Hom}(\mathcal{E}, \mathcal{G})$ in \mathcal{EQ} .

- (i) If f is injective, then f is monic.
- (ii) If \mathcal{E} is a separated EQ-algebra and f is monic, then f is an injective map.
- (iii) If f is onto, then f is epic.

PROOF: (i) The proof is clear.

(ii) Let $H = \{a \in E \mid f(a) = 1_{\mathcal{G}}\}$. It is easy to see that $\mathcal{H} = (H, \otimes_{\mathcal{H}}, \wedge, \sim_{\mathcal{H}}, 1_{\mathcal{E}})$ is a sub-algebra of \mathcal{E} . Suppose $i : \mathcal{H} \rightarrow \mathcal{E}$ is an inclusion morphism and $g : \mathcal{H} \rightarrow \mathcal{E}$ is an EQ-homomorphism such that for any $a \in H$, $g(a) = 1_{\mathcal{E}}$. Since f is monic if $f \circ i = f \circ g$, then $H = \{1\}$. Since \mathcal{E} is separated, by Proposition 3.2, f is injective.

(iii) Suppose that $g, h : \mathcal{G} \rightarrow \mathcal{H}$ are two morphisms such that $g \circ f = h \circ f$. Since f is onto, for any $b \in G$, there is $a \in E$ where $f(a) = b$. Thus, for any $b \in G$,

$$g(b) = g(f(a)) = g \circ f(a) = h \circ f(a) = h(f(a)) = h(b).$$

Hence f is epic. □

In the following example we show that the converse of Theorem 3.5 (iii) is not true, in general.

Example 3.6. Let $E = \{0, a, b, c, d, 1\}$ be a lattice with a Hesse diagram as Figure 1. For any $x, y \in E$, we define the operations \otimes and \sim on E as Table 5 and Table 6:

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is a good EQ-algebra. Let \mathcal{H} be the EQ-algebra as in Example 3.3. Then the map $f : H \rightarrow E$ where $f(0) = 0$, $f(a) = b$, $f(b) = a$ and $f(1) = 1$ is a non-onto homomorphism. Let $id : E \rightarrow E$ be the identity map and $t : E \rightarrow E$ be the trivial EQ-homomorphism. It is clear that $h \neq g$. Since $id \circ f(a) = b$ and $t \circ f(a) = 1$, we get $h \circ f \neq g \circ f$.

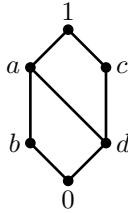


Figure 1

Table 5

\otimes	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	b	d	0	a
b	0	b	b	0	0	b
c	0	d	0	c	d	c
d	0	0	0	d	0	d
1	0	a	b	c	d	1

Table 6

\sim	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	a	c	d	a	1	d
1	0	a	b	c	d	1

3.2. Objects

In this subsection, we show that \mathcal{EQ} has zero objects and introduce the free EQ -algebra on the singleton.

THEOREM 3.7. *In \mathcal{EQ} , $\{1\}$ is the zero object.*

PROOF: Let \mathcal{E} be an arbitrary EQ -algebra. Then $f : \{1\} \rightarrow \mathcal{E}$ where $f(1) = 1_{\mathcal{E}}$ is an EQ -homomorphism. On the other hand, let $g : \mathcal{E} \rightarrow \{1\}$ be a map where for any $a \in E$, $g(a) = 1$. Since g preserves the operations \otimes, \wedge and \sim , we obtain that g is an EQ -homomorphism. \square

COROLLARY 3.8. The zero objects are the only injective object in \mathcal{EQ} .

In [14], Novák and El-zekey proved that the class of EQ -algebras is a variety. Thus, by Theorem 2.11, there exists a free EQ -algebra on a given set.

In [3], Blyth showed that the free monoid on a singleton is isomorphic to the additive monoid \mathbb{Z}^+ . So we can consider free monoid on element $\{x\}$, by (\mathcal{E}_x, \otimes) where $\mathcal{E}_x = \{e = x^0, x^1, x^2, \dots, x^i, \dots\}$. For any $i \in \mathbb{N}$,

$x^i \otimes x^j = x^{i+j}$ and e is the identity element. Now, we define a relation on \mathcal{E}_x as follows: for any $x^i, x^j \in \mathcal{E}_x$, we say that $x^i \leq_f x^j$ if and only if $j \leq i$.

LEMMA 3.9. *Let (\mathcal{E}_x, \otimes) be a free monoid on $\{x\}$.*

- (i) *The relation \leq_f on \mathcal{E}_x is an order.*
- (ii) *For any $i, j, k \in \mathbb{Z}^+$, if $x^i \leq_f x^j$, then $x^i \otimes x^k \leq_f x^j \otimes x^k$.*

PROOF: (i) We show that \leq_f is reflexive, antisymmetric, and transitive. Since \mathbb{Z} is an ordered set, for any $i \in \mathbb{Z}^+$, we have $x^i \leq_f x^i$ and \leq_f is reflexive.

Suppose that for $i, j \in \mathbb{N}$, $x^i \leq_f x^j$ and $x^j \leq_f x^i$. By definition of \leq_f , we obtain $j \leq i$ and $i \leq j$. Thus $i = j$, then $x^i = x^j$ and so \leq_f is antisymmetric.

Suppose for some $i, j, k \in \mathbb{N}$, $x^i \leq_f x^j$ and $x^j \leq_f x^k$. By definition of \leq_f , we have $j \leq i$ and $k \leq j$. Hence $k \leq i$, $x^i \leq_f x^k$ and so \leq_f is transitive.

Since \mathbb{N} is a chain, we can see that $\mathcal{E} = (E, \leq_f)$ is a chain with maximum element. Therefore, it is meet semilattice with upper bound.

- (ii) Now suppose that $x^i \leq_f x^j$, then $j \leq i$. For any $k \in \mathbb{Z}^+$, $j+k \leq i+k$ and so $x^{i+k} \leq_f x^{j+k}$. Thus, $x^i \otimes x^k \leq_f x^j \otimes x^k$. Therefore, \leq_f is an order relation on \mathcal{E} . □

THEOREM 3.10. *Let $X = \{x\}$ be a set and $\mathcal{E} = (E, \otimes, \wedge)$ be a free monoid on X with an order we define in Lemma 3.9. If for any $i, j \in \mathbb{Z}^+$, we define a fuzzy equality on \mathcal{E} as $x^i \sim x^j = x^{|i-j|}$, then $\mathcal{E}_x = (E, \otimes, \wedge, \sim, e)$ is an EQ-algebra.*

PROOF: By Lemma 3.9, (E1) and (E2) are satisfied. For any $i \in \mathbb{Z}^+$, $x^i \sim x^i = x^{|i-i|} = x^0 = e$, and so (E3) is satisfied. Without loss of generality, in the rest of proof we suppose that $i \leq j \leq k \leq w$. Then

$$((x^i \wedge x^j) \sim x^k) \otimes (x^w \sim x^i) = (x^j \sim x^k) \otimes (x^w \sim x^i) = x^{w+k-(i+j)}.$$

On the other hand, $x^k \sim (x^j \wedge x^w) = x^{w-k}$. Moreover, since $i, j \leq k$, we have $i+j \leq 2k$ and so $w-k \leq w+k-(i+j)$. Thus, $x^{w+k-(i+j)} \leq_f x^{w-k}$. Hence (E4) holds.

We can see that $(x^i \sim x^j) \otimes (x^k \sim x^w) = x^{w+j-(i+k)}$ and $(x^i \sim x^k) \sim (x^j \sim x^w) = x^{|(w-j)-(k-i)|}$. To show that (E5) is satisfied, we consider two following cases.

Case 1. If $k + j \leq w + i$, then $k + j - (w + i) \leq w + j - (i + k)$ and so

$$(x^i \sim x^j) \otimes (x^k \sim x^w) \leq (x^i \sim x^k) \sim (x^j \sim x^w).$$

Case 2. If $w + i \leq k + j$, then $x^{w+j-(i+k)} \leq x^{(w-j)-(k-i)}$ and so (E5) holds.

Moreover, since $j - i \leq k - i$, we get that $x^{k-i} \leq x^{j-i}$ and so $(x^i \wedge x^j \wedge x^k) \sim x^i \leq (x^i \wedge x^j) \sim x^i$. Thus, (E6) is satisfied.

Also, from $i, j \leq j + i$, we have $x^i \otimes x^j \leq x^i \wedge x^j$, and (E7) is satisfied. Therefore, $\mathcal{E}_x = (E, \otimes, \wedge, \sim, e)$ is an *EQ*-algebra. □

Remark 3.11. For any $i, j \in \mathbb{Z}^+$, the implication operation on \mathcal{E}_x is

$$x^i \rightarrow x^j = \begin{cases} e & j \leq i \\ x^{j-i} & i < j. \end{cases}$$

PROPOSITION 3.12. Let $X = \{x\}$ be a set. Then the following statements hold:

- (i) \mathcal{E}_x is a good *EQ*-algebra.
- (ii) \mathcal{E}_x is a residuated *EQ*-algebra.
- (iii) \mathcal{E}_x is an *ℓEQ*-algebra.
- (iv) \mathcal{E}_x is a multiplicatively relative *EQ*-algebra.

PROOF: (i) By the definition of \sim on \mathcal{E}_x , the proof is clear.

(ii) Let $i, j, w \in \mathbb{Z}^+$. If $w \leq i + j$, then $(x^i \otimes x^j) \rightarrow x^w = e$. Thus, we consider two following cases.

Case 1. If $w \leq j$, then $x^j \rightarrow x^w = e$ and so for any $i \in \mathbb{Z}^+$, we have

$$(x^i \otimes x^j) \rightarrow x^w = x^i \rightarrow (x^j \rightarrow x^w) = e.$$

Case 2. If $j < w$, then $x^j \rightarrow x^w = x^{w-j}$ and since $w - j \leq i$ we have $x^i \rightarrow (x^j \rightarrow x^w) = e$.

Now, if $i + j < w$, then $(x^i \otimes x^j) \rightarrow x^w = x^{w-(i+j)}$. Moreover, since $i < w$ and $j < w$, we get that $x^i \rightarrow (x^j \rightarrow x^w) = x^{w-j-i}$. Hence, by Proposition 2.1 (ii), \mathcal{E}_x is residuated.

(iii) Without loss of generality, we consider that $i \leq j \leq k \leq w$. Since $2i \leq w + j$, we have

$$((x^i \vee x^j) \sim x^k) \otimes (x^w \sim x^i) \leq ((x^j \vee x^w) \sim x^k).$$

(iv) By Lemma 3.9, we know that \mathcal{E}_x is a chain. Then by Proposition 2.2, it is multiplicatively relative. \square

Remark 3.13. Since \mathcal{E}_x does not have least element, it is not a residuated lattice.

In the following example, we show that \mathcal{E}_x is not free in general.

Example 3.14. Let \mathcal{E} be an EQ-algebra as in Example 3.6. Let $X = \{x\}$ be an arbitrary set and $h : X \rightarrow E$ be a map where $h(x) = b$. If \mathcal{E}_x is a free EQ-algebra, then $f : \mathcal{E}_x \rightarrow \mathcal{E}$ is an EQ-homomorphism which $f(x) = h(x)$ and so for any $i \in \mathbb{N}$, we should have $f(x^i) = h(x) \otimes h(x) \otimes \dots \otimes h(x) = b$. We claim that f is not an EQ-algebra homomorphism. Because, for $j > i$, $f(x^i \sim x^j) = f(x^{j-i}) = b$ but, $f(x^i) \sim f(x^j) = b \sim b = 1 \neq b$. Thus, f is not preserves the both operations \otimes and \sim at the same time.

Remark 3.15. Let $X = \{x\}$ be a set and \mathcal{E}_x be the EQ-algebra as in Theorem 3.10. For any $i, j, k \in \mathbb{Z}^+$, $x^i \sim x^j = x^{i+k} \sim x^{j+k} = (x^i \otimes x^j) \sim (x^j \otimes x^k)$.

DEFINITION 3.16. An EQ-algebra \mathcal{E} is *multiplicatively equal* if for any $a, b, c \in E$,

$$a \sim b = (a \otimes c) \sim (b \otimes c).$$

Example 3.17. By Remark 3.15, \mathcal{E}_x is multiplicatively equal.

THEOREM 3.18. *The EQ-algebra \mathcal{E}_x is a free object on the class of good multiplicatively equal EQ-algebras.*

PROOF: Let $\mathcal{H} = (H, \otimes_H, \wedge_H, \sim_H, e_H)$ be a multiplicatively equal EQ-algebra, $X = \{x\}$ be a set and $g : X \rightarrow H$ be a map. We define a map $f : \mathcal{E}_x \rightarrow H$ such that for any $i \in \mathbb{N}$, $f(x^i) = g(x)^i$ and $f(e_{\mathcal{E}_x}) = e_H$. Now, we show that f is an EQ-homomorphism. Let $i, j \in \mathbb{N}$,

$$f(x^i \otimes x^j) = f(x^{i+j}) = g(x)^{i+j} = g(x)^i \otimes_H g(x)^j.$$

Without loss of generality, we can consider that $i \leq j$ and so $g(x)^j = g(x)^i \otimes g(x)^{j-i}$. By (E7), we have $g(x)^i \wedge_H g(x)^j = g(x)^j$. Thus, $f(x^i \wedge x^j) = f(x^j) = g(x)^j = g(x)^i \wedge g(x)^j$.

Since \mathcal{H} is good multiplicatively equal, for any $j > i \in \mathbb{N}$, we get that,

$$g(x)^{j-i} = e_H \sim g(x)^{j-i} = g(x)^i \sim g(x)^j = f(x^i) \sim f(x^j).$$

On the other hand, $f(x^i \sim x^j) = f(x^{j-i}) = g(x)^{j-i}$ and so f preserves the operation " \sim ". Therefore, f is an EQ -homomorphism. \square

3.3. Limits

In this subsection, we show \mathcal{EQ} has products and also all finite limits.

THEOREM 3.19. *\mathcal{EQ} has*

- (i) *product,*
- (ii) *equalizers,*
- (iii) *pullbacks,*
- (iv) *all finite limits,*
- (v) *finite intersections.*

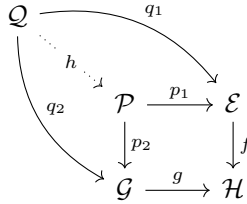
PROOF: (i) Since the class of EQ -algebras is a variety, by Theorem 2.10, it has products. Then for any EQ -algebras \mathcal{E} and \mathcal{G} , $\mathcal{E} \times \mathcal{G}$ with pointwise operations is an EQ -algebra. Thus, $\mathcal{E} \times \mathcal{G}$ with projection maps (p_1, p_2) is the product of \mathcal{E} and \mathcal{G} .

(ii) Let $f, g : \mathcal{E} \rightarrow \mathcal{G}$ be two EQ -homomorphisms and let $H = \{a \in \mathcal{E} \mid f(a) = g(a)\}$. Since f, g are homomorphisms, $\mathcal{H} = (H, \otimes_{\mathcal{E}}, \wedge_{\mathcal{E}}, \sim_{\mathcal{E}}, 1_{\mathcal{E}})$ is an EQ -algebra. Let $i : \mathcal{H} \rightarrow \mathcal{E}$ be the inclusion map. Then for any $a \in H$, $f \circ i(a) = f(a) = g(a) = g \circ i(a)$.

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{i} & \mathcal{E} \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & \mathcal{G} \\ \uparrow i & \nearrow j & & \\ \mathcal{K} & & & \end{array}$$

Now, suppose that \mathcal{K} is an EQ -algebra and $j : \mathcal{K} \rightarrow \mathcal{E}$ is a morphism such that $f \circ j = g \circ j$. Then for any $x \in K$, $j(x) \in H$ and $Im(j) \subseteq H$. Thus, we can define a morphism $l : \mathcal{K} \rightarrow \mathcal{H}$ where for any $x \in K$, $l(x) = j(x)$. It is clear that l is an unique EQ -homomorphism and $i \circ l = j$. Hence (\mathcal{H}, i) is the equalizer of f, g .

(iii) Let $f \in \text{Hom}(\mathcal{E}, \mathcal{H})$ and $g \in \text{Hom}(\mathcal{G}, \mathcal{H})$. Since \mathcal{EQ} has a binary product, $\mathcal{P} = \{(a, b) \in E \times G \mid f(a) = g(b)\}$ is an EQ -algebra and so (\mathcal{P}, p_1, p_2) is the pull back of (f, g) .



Now suppose that (\mathcal{Q}, q_1, q_2) is an EQ -algebra with two morphisms where $f \circ q_1 = g \circ q_2$. Let $h : \mathcal{Q} \rightarrow \mathcal{P}$ be a map where for any $x \in \mathcal{Q}$, $h(x) = (q_1(x), q_2(x))$. Since q_1 and q_2 are homomorphisms, h is homomorphism. By considering the definition of h , we can see that $p_1 \circ h = q_1$ and $p_2 \circ h = q_2$. Moreover, since p_1 and p_2 are onto, they are epic and so h is unique.

(iv), (v) Since \mathcal{EQ} has all finite products and equalizers, by Theorem 2.9, it has all finite limits. □

3.4. Co-limits

In this subsection, we show that \mathcal{EQ} does not have co-limits such as co-equalizers, coproduct and pushout, in general. In the rest of this article, we introduce a method to extend any good EQ -algebra and by using this method we calculate coproducts and push out of EQ -algebras in special cases.

THEOREM 3.20. *Let $f, g : \mathcal{E} \rightarrow \mathcal{G}$ be two EQ -homomorphisms. If \mathcal{G} is a multiplicatively relative EQ -algebra, then f, g have co-equalizer.*

PROOF: Let

$$\mathcal{F} = \{F \mid F \text{ is a filter of } \mathcal{G} \text{ such that for any } x \in E, f(x) \sim g(x) \in F\}.$$

Since \mathcal{G} is a multiplicatively relative EQ -algebra, $\langle \text{Im}f \cup \text{Im}g \rangle$ is a filter of \mathcal{G} and so \mathcal{F} is not empty. By Proposition 2.5, $\cap \mathcal{F}$ is a filter of \mathcal{G} . Then by Theorem 2.6, $\mathcal{G}/\cap \mathcal{F}$ is an EQ -algebra. Let $\pi : \mathcal{G} \rightarrow \mathcal{G}/\cap \mathcal{F}$ be a map such that for any $a \in G$, $\pi(a) = [a]$. For any $x \in E$, $\pi \circ f(x) = [f(x)]$ and $\pi \circ g(x) = [g(x)]$. Since $f(x) \sim g(x) \in \cap \mathcal{F}$, we have $[f(x)] = [g(x)]$.

Suppose that there exists a separated EQ -algebra and a homomorphism such as (\mathcal{J}, j) where $j \circ f = j \circ g$. Since, for any $x \in E$, $j(f(x) \sim g(x)) = (j \circ f(x)) \sim (j \circ g(x)) = 1$, we get that $f(x) \sim g(x) \in \ker j$.

Now, let $k : \mathcal{G} / \cap \mathcal{F} \rightarrow \mathcal{J}$ be a map such that $k([a]) = j(a)$ for any $[a] \in \mathcal{G} / \cap \mathcal{F}$. We show that k is a homomorphism.

$$\begin{array}{ccc}
 \mathcal{E} & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & \mathcal{G} & \xrightarrow{\pi} & \frac{\mathcal{G}}{\cap \mathcal{F}} \\
 & & \searrow j & & \downarrow k \\
 & & & & \mathcal{J}
 \end{array}$$

Suppose that $[a] = [b]$. By Proposition 2.6 we have, $a \sim b \in \cap \mathcal{F}$. By Proposition 2.4, $\ker j$ is a filter of \mathcal{G} and so $j(a) \sim j(b) = 1$. Since \mathcal{J} is separated, $j(a) = j(b)$ and so $k([a]) = k([b])$.

By considering the definition of k , it is clear that $k \circ \pi = j$ and k is an unique EQ -homomorphism. Therefore, $(\mathcal{G} / \cap \mathcal{F}, \pi)$ is a co-equalizer. \square

THEOREM 3.21. *Let \mathcal{E} be a good. If $e \notin E$, then $\mathcal{E}' = (E \cup \{e\}, \otimes_{\mathcal{E}'}, \wedge_{\mathcal{E}'}, \sim_{\mathcal{E}'}, e)$ is a good EQ -algebra where $\otimes_{\mathcal{E}'}, \wedge_{\mathcal{E}'}$, and $\sim_{\mathcal{E}'}$ define as follows:*

$$a \otimes_{\mathcal{E}'} b = \begin{cases} a \otimes b & a, b \in E \\ a & a \in E, b = e \\ b & b \in E, a = e \\ e & a = b = e. \end{cases} \quad a \wedge_{\mathcal{E}'} b = \begin{cases} a \wedge b & a, b \in E \\ a & a \in E, b = e \\ b & b \in E, a = e \\ e & a = b = e. \end{cases}$$

$$a \sim_{\mathcal{E}'} b = \begin{cases} a \sim b & a, b \in E, a \neq b \\ e & a, b \in E, a = b \\ a & a \in E, b = e \\ b & b \in E, a = e \\ e & a = b = e. \end{cases}$$

PROOF: By considering the definition of $\otimes_{\mathcal{E}'}$, we can see that $(\mathcal{E}', \otimes_{\mathcal{E}'}, e)$ is a commutative monoid [4]. Also, $(\mathcal{E}', \wedge_{\mathcal{E}'}, e)$ is a meet semilattice with upper bound e . Now, we show $\otimes_{\mathcal{E}'}$ is isotone with respect to $\leq_{\mathcal{E}'}$. Let $a, b \in E \cup \{e\}$ such that $a \leq_{\mathcal{E}'} b$. We can consider two cases.

Case 1. We suppose that $a \leq b < e$. If $c \in E$, then $a \otimes_{\mathcal{E}'} c = a \otimes c \leq b \otimes c = b \otimes_{\mathcal{E}'} c$. If $c = e$, then $a \otimes_{\mathcal{E}'} e = a \leq b = b \otimes_{\mathcal{E}'} e$.

Case 2. Since for any $a \in E$, $a \leq_{\mathcal{E}'} e$, we have $a \otimes_{\mathcal{E}'} c = a \otimes c \leq c = e \otimes_{\mathcal{E}'} c$, for any $c \in E$. If $c = e$, then $a = a \otimes_{\mathcal{E}'} e \leq_{\mathcal{E}'} e = e \otimes_{\mathcal{E}'} e$. Thus, (E2) holds.

By considering the definition of $\sim_{\mathcal{E}'}$, we can see that (E3) is satisfied. To show that (E4) is satisfied on \mathcal{E}' we can consider four following cases.

Case 1. Suppose that $a, b, c \in E$. Since \mathcal{E} is good, we have,

$$\begin{aligned} ((a \wedge b) \sim_{\mathcal{E}'} e) \otimes_{\mathcal{E}'} (c \sim a) &= (a \wedge b) \otimes (c \sim a) \\ &= ((a \wedge b) \sim 1) \otimes (c \sim a) \\ &\leq (1 \sim (c \wedge b)) \\ &= c \wedge b \\ &= (e \sim_{\mathcal{E}'} (c \wedge_{\mathcal{E}'} b)). \end{aligned}$$

Case 2. Since \mathcal{E} is an EQ -algebra, by (E5) we have,

$$\begin{aligned} ((a \wedge_{\mathcal{E}'} e) \sim_{\mathcal{E}'} c) \otimes_{\mathcal{E}'} (b \sim_{\mathcal{E}'} a) &= (a \sim c) \otimes (b \sim a) \\ &\leq (a \sim a) \sim (c \sim b) \\ &= 1 \sim (c \sim b) \\ &= (c \sim_{\mathcal{E}'} (b \wedge_{\mathcal{E}'} e)). \end{aligned}$$

Case 3. Since \mathcal{E} is good, we have

$$\begin{aligned} ((a \wedge_{\mathcal{E}'} b) \sim_{\mathcal{E}'} c) \otimes_{\mathcal{E}'} (e \sim_{\mathcal{E}'} a) &= ((a \wedge b) \sim c) \otimes (1 \sim a) \\ &\leq (c \sim (1 \wedge b)) \\ &= (c \sim_{\mathcal{E}'} (e \wedge_{\mathcal{E}'} b)). \end{aligned}$$

Case 4. Since \mathcal{E} is good by Proposition 2.3 (ii), we have,

$$\begin{aligned} ((a \wedge_{\mathcal{E}'} e) \sim_{\mathcal{E}'} e) \otimes_{\mathcal{E}'} (d \sim_{\mathcal{E}'} a) &= a \otimes (d \sim a) \leq d \\ &= e \sim_{\mathcal{E}'} (d \wedge_{\mathcal{E}'} e). \end{aligned}$$

For any $a, b, c, d \in E$, (E5) is satisfied. Now, we show that (E6) holds.

$$\begin{aligned}
 (a \sim_{\mathcal{E}'} b) \otimes_{\mathcal{E}'} (c \sim_{\mathcal{E}} e) &= (a \sim b) \otimes (c \sim 1) \\
 &\leq (a \sim c) \sim (b \sim 1) \\
 &= (a \sim_{\mathcal{E}'} c) \sim_{\mathcal{E}'} (b \sim_{\mathcal{E}'} e).
 \end{aligned}$$

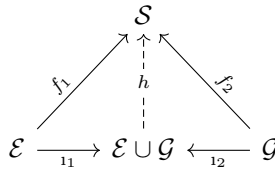
Since for any $a, b, c \in E$, $(a \wedge_{\mathcal{E}'} b \wedge_{\mathcal{E}'} e) \sim_{\mathcal{E}'} a = (a \wedge_{\mathcal{E}'} b) \sim_{\mathcal{E}'} a$ and $(e \wedge_{\mathcal{E}'} b \wedge_{\mathcal{E}'} c) \sim_{\mathcal{E}'} e = (b \wedge_{\mathcal{E}'} c) \sim_{\mathcal{E}'} e$, (E6) is satisfied. For any $a \in E$, $a \otimes_{\mathcal{E}'} e = a = a \sim_{\mathcal{E}'} e$, and so (E7) is satisfied. \square

COROLLARY 3.22. Let X be a countable chain with maximum element x . If \mathcal{E} is a good EQ -algebra, then $\mathcal{E}' = (E \cup X, \otimes_{\mathcal{E}'}, \wedge_{\mathcal{E}'}, \sim_{\mathcal{E}'}, x)$ is a good EQ -algebra.

PROOF: By induction on the cardinality of X and Theorem 3.21, the proof is clear. \square

THEOREM 3.23. Let \mathcal{E} be a good EQ -algebra and $\mathcal{G} = \{e\}$. Then \mathcal{E} and \mathcal{G} have co-product.

PROOF: By Theorem 3.21, $\mathcal{E} \cup \mathcal{G}$ is an EQ -algebra. Let $i_2 : \mathcal{G} \rightarrow \mathcal{E} \cup \mathcal{G}$ be the inclusion map and $i_1 : \mathcal{E} \rightarrow \mathcal{E} \cup \mathcal{G}$ be a map such that $i_1(1) = e$ and for any $a \in E - \{1\}$, $i_1(a) = a$. We claim that $(\mathcal{E} \cup \mathcal{G}, i_1, i_2)$ is the co-product of \mathcal{E} and \mathcal{G} .



Suppose that (\mathcal{S}, f_1, f_2) is an EQ -algebra with two homomorphisms such that $f_1 : \mathcal{E} \rightarrow \mathcal{S}$ and $f_2 : \mathcal{G} \rightarrow \mathcal{S}$. We define a map $h : \mathcal{E} \cup \mathcal{G} \rightarrow \mathcal{S}$ as follows:

$$h(a) = \begin{cases} f_1(a) & a \in E \\ f_2(a) & a \in G. \end{cases}$$

Since f_1 and f_2 are EQ -homomorphisms, h is an EQ -homomorphism and by definition of h we can see that $h \circ i_1 = f_1$ and $h \circ i_2 = f_2$ and so, h is unique. \square

In the following example, we show that \mathcal{EQ} does not have co-product, in general.

Example 3.24. Let $H = \{d, e_{\mathcal{H}}\}$ be an EQ -algebra and $E = \{0, a, b, 1\}$ be a chain where $0 \leq a \leq b \leq 1$. For any $\alpha, \beta \in E$, we define the operations \otimes and \sim on E as Table 7 and Table 8:

Table 7

\otimes	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	0	b
1	0	a	b	1

Table 8

\sim	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	b
1	0	a	b	1

By routine calculations, we can see that $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is a good EQ -algebra. Let $G = \{0, x, 1\}$ be a chain where $0 \leq x \leq 1$. For any $\alpha, \beta \in E$, we define the operations \otimes and \sim on E as Table 9 and Table 10.

Table 9

\otimes	0	x	1
0	0	0	0
x	0	0	x
1	0	x	1

Table 10

\sim	0	x	1
0	1	0	0
x	0	1	x
1	0	x	1

We can see that $\mathcal{G} = (G, \wedge, \otimes, \sim, 1)$ is a good EQ -algebra. By routine calculations we can see that $f_1 : E \rightarrow G$ such that $f_1(0) = 0$, $f_1(a) = x$, and $f_1(b) = f_1(1) = 1$ is an EQ -homomorphism. By Corollary 3.22, $\mathcal{E} \cup \mathcal{H}$ is an EQ -algebra. Suppose that $f_2 : \mathcal{H} \rightarrow \mathcal{G}$ be a map where $f_2(d) = 0$ and $f_2(e_{\mathcal{H}}) = 1_{\mathcal{G}}$. If $(\mathcal{E} \cup \mathcal{H}, i_1, i_2)$ is the co-product of \mathcal{E} and \mathcal{H} , there exists a homomorphism $h : \mathcal{E} \cup \mathcal{H} \rightarrow \mathcal{G}$ such that $i_2 \circ h = f_2$ and $i_1 \circ h = f_1$. Since i_2 is a homomorphism we get that $i_2(e_{\mathcal{H}}) = 1_{\mathcal{G}}$. Now, we consider three cases for $i_2(d)$.

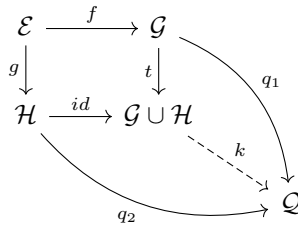
Case 1. Suppose that $i_2(d) = 0$. Then we should have $h \circ i_2(d) = h(0) = f_2(d) = 0$. By Theorem 3.21, $a \otimes d = a$ and so we should have $h(a \otimes d) = h(a) \otimes h(d) = h(a) \otimes 0 = h(a)$. According to Table 12, $h(a) = 0$. On the other hand $h(0) \sim h(a) = 1_{\mathcal{G}}$ and $h(0 \sim a) = h(0) = 0$, which means that h is not a homomorphism.

Case 2. If $i_2(d) \in \{a, b, d, 1\}$, then $h(0) = h(a) = 0$ with similar way in Case 1, we can see that h is not homomorphism.

Case 3. Suppose that $i_2(d) = e_{\mathcal{H}}$. Then $h \circ i_2(d) = h(e_{\mathcal{H}}) = f_2(d) = 0$.
 But if h is a homomorphism, then we should have $h(e_{\mathcal{H}}) = 1_{\mathcal{G}}$.

THEOREM 3.25. *Let \mathcal{G} be a good EQ-algebra and $\mathcal{H} = \{e\}$. If $f : \mathcal{E} \rightarrow \mathcal{G}$ and $g : \mathcal{E} \rightarrow \mathcal{H}$ are an arbitrary and trivial EQ-homomorphisms, respectively, then (\mathcal{E}, f, g) has pushout.*

PROOF: By Theorem 3.23, $(\mathcal{G} \cup \mathcal{H})$ is the co-product of \mathcal{G} and \mathcal{H} . Let $t : \mathcal{G} \rightarrow \mathcal{G} \cup \mathcal{H}$ be the trivial homomorphism. For any $a \in E$, we have $t \circ f(a) = t(f(a)) = e_{\mathcal{H}} = id \circ g(a)$ and the following diagram is commutative.



Suppose that (\mathcal{Q}, q_1, q_2) is an EQ-algebra with two homomorphisms where $q_1 \circ f = q_2 \circ g$. We can see that q_1 is trivial EQ-homomorphism, too. If $k : \mathcal{G} \cup \mathcal{H} \rightarrow \mathcal{Q}$ is the trivial map, then the above diagram is commutative and also k is unique. □

4. Conclusions and future works

In this paper, the category of EQ-algebras is studied and showed that it is complete, but it is not cocomplete, in general. It is proved that the multiplicatively relative EQ-algebras have coequalizers and coproduct and pushout in a special case. Also, the free EQ-algebra on a singleton is constructed. Since every good EQ-algebra is an equality algebra [20], most results of this paper hold for equality algebras, too. For the future work, we can find free EQ-algebra on any set. Maybe, there exist some special kind of EQ-algebras which have co-product and pushout in general.

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